On L^1-stability of BV solutions for a model of granular flow

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Abstract

We are concerned with the well-posedness of a model of granular flow that consists of a hyperbolic system of two balance laws in one-space dimension, which is linearly degenerate along two straight lines in the phase plane and genuinely nonlinear in the subdomains confined by such lines. This note provides a survey of recent results [3] on the Lipschitz $\mathbf{L^1}$-continuous dependence of the entropy weak solutions on the initial data, with a Lipschitz constant that grows exponentially in time. Our analysis relies on the extension of a Lyapunov like functional and provide the first construction of a Lipschitz semigroup of entropy weak solutions to the regime of hyperbolic systems of balance laws (i) with characteristic families that are neither genuinely nonlinear nor linearly degenerate and (ii) initial data of arbitrarily large total variation.

1 Introduction

We consider the system of balance laws

$$\begin{aligned} h_t - (hp)_x &= (p-1)h, \\ p_t + ((p-1)h)_x &= 0, \end{aligned} \tag{1.1}$$

with $h \geq 0$ and $p \geq 0$. System (1.1) represents the model in the one space dimensional setting proposed by Hadeler and Kuttler [12] for the flow of granular material and describes he evolution of a moving layer on top and of a resting layer at the bottom. Here, the unknown $h = h(x,t)$ and $p(x,t)$ represent, respectively, the thickness of the rolling layer and the slope of the standing layer, while $t \geq 0$ and $x \in \mathbb{R}$ are the time and space variables. The evolution equations (1.1) show that the moving layer slides downhill with speed proportional to the slope of the standing layer in the direction of steepest descent. The model (1.1) is written in normalised form, assuming that the critical slope is $p = 1$. This means that, if $p > 1$, then grains initially at rest are hit by rolling matter of the moving layer and hence they start moving too; thus, the moving layer gets thicker. On the other hand, if $p < 1$, then rolling grains can be deposited on the standing bed and, hence, the moving layer becomes thinner. Typical examples of granular material whose dynamics is described by such models are dry sand and gravel in dunes and heaps, or snow in avalanches.

This article serves as a survey of the analysis in [3] on the well-posedness of the Cauchy problem for (1.1). More precisely, in [3], the authors obtain a Lipschitz continuous semigroup of entropy weak solutions to the nonlinear system of balance laws (1.1) via a Lyapunov type functional with large initial data. Besides the motivation of this analysis in the setting of the granular flow model, the results provide the first construction of a semigroup for

(i) systems with characteristic families that are neither genuine nonlinear (GNL) nor linear degenerate (LD) (nor of Temple class), and

(ii) initial data with arbitrary large total variation.

The aim here is to provide a short exposition on the analysis of [3] pointing out the challenges that arise by these features and comparing the Lyapunov functional introduced in [3] with the classical one of Bressan et al [9].

Since, in general, global smooth solutions to hyperbolic systems do not exist, we consider weak solutions in the sense of distributions and in particular, an *entropy-admissible weak solution* of (1.1), that means a weak solution, admissible in the sense of Lax. Global existence of classical smooth solutions to (1.1) were established for a special class of initial data by Shen [14]. In the case of more general initial data with bounded but possibly large total variation, the existence of entropy weak solutions globally defined in time was proved by Amadori and Shen [2].

For systems without source term and small BV data, the Lipschitz $\mathbf{L^1}$-continuous dependence of solutions on the initial data, was first established by Bressan and collaborators in [7, 8] under the assumptions that all characteristic families are genuinely nonlinear (GNL) or linearly degenerate (LD), relying on a homotopy method that is lengthy and involves several technical points. An extension of these results is established in [4] to a class of 2×2 systems

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with non GNL characteristic fields that does not comprise the convective part of system (1.1). A different proof of the $\mathbf{L}^1$-stability of solutions for conservation laws with GNL or LD characteristic fields that is less technical and more transparent was later achieved by a technique introduced by Liu and Yang in [13] and then developed by Bressan et al [9]. Extensions of $\mathbf{L}^1$-stability results to the setting of large BV data was obtained for systems of conservation laws with Temple type characteristic fields and other special systems and also for balance laws with small data. A rich bibliography on these references can be found in [3] as well as further ones on other models of granular flow.

However, our system (1.1) does not fulfill these classical assumptions and in addition, its special source terms do not belong within a class for which L^1 stability results are available in the literature. The heart of the matter in [3] is to construct a Lyapunov-like nonlinear functional Φ, equivalent to the $\mathbf{L}^1$-distance, which is decreasing in time along any pair of solutions. In this review article, we state some preliminary results in Section 2, and then present the stability functional Φ in Section 3 comparing it with the classical one of Bressan et al [9] and providing the motivation of our construction. In Section 4, we conclude stating our main theorems and refering to [3] for the proofs and further analysis.

2 Preliminaries

It is easy to verify that system (1.1) is strictly hyperbolic on the domain

$$\Omega \doteq \{(h,p) : h \geq 0,\ p > 0\} \tag{2.1}$$

and weakly linearly degenerate at the point $(h,p) = (0,1)$. We observe that the line $p = 1$ separates the domain Ω into two invariant regions for solutions of the Riemann problem: the quarter $\{h \geq 0,\ p > 1\}$ and the half-strip $\{h \geq 0,\ 0 < p < 1\}$. Indeed, the rarefaction and Hugoniot curves of the first family through a point (h_ℓ, p_ℓ), with $p_\ell \neq 1$, never meets the line $p = 1$, while the rarefaction and Hugoniot curves of the second family through a point (h_ℓ, p_ℓ), with $h_\ell > 0$, never meets the line $h = 0$. On the the other hand, the lines $p = 1$ and $h = 0$ are also invariant regions for solutions of the Riemann problem since they coincide with the rarefaction and Hugoniot curves of the first and second family, respectively, passing through any of their points. Notice that, although the characteristic field of the first family does not satisfy the classical GNL assumption, no composite waves are present in the solution of a Riemann problem for

$$\begin{aligned} &h_t - (hp)_x = 0, \\ &p_t + ((p-1)h)_x = 0, \end{aligned} \tag{2.2}$$

since in each invariant region $\{p > 1\}$, $\{p < 1\}$ the field is GNL. In fact, the general solution of a Riemann problem for (2.2) consists of at most one simple wave for each family which can be either a rarefaction or a compressive shock or a contact discontinuity.

Let $u = u(x,t) \doteq (h^{s,\varepsilon}, p^{s,\varepsilon})(x,t)$ be a piecewise constant s-ε-approximate solution converging to an entropy weak solutions to (1.1) with initial data

$$h(x,0) = \overline{h}(x)\,, \qquad p(x,0) = \overline{p}(x) \qquad \text{for a.e. } x \in \mathbb{R}\,. \tag{2.3}$$

constructed as in [2] by the usual operator splitting scheme as $\varepsilon \to 0+$ and $s \to 0+$. Here, $s = \Delta t > 0$ stands for the time step and a parameter $\varepsilon > 0$ a small positive parameter of the front tracking algorithm. We refer to [11] and [1] for the early works on this subject and also point out that the source term (1.1) does not belong in the class of the so-called "dissipative" terms exploited in [11, 1, 10]. As usual, a-priori bounds on the total variation of $u(t) \doteq u(\cdot,t)$ outside the time steps are obtained in [2] by analyzing suitable wave strength and wave interaction potential that are defined as follows.

First, the sizes of wave fronts of approximate solutions of (1.1) are defined as the jumps between the left and right states either measured with the original variables (h,p) or with the corresponding Riemann coordinates (H,P) associated to system (1.1). So given a wave front with left and right states (h_ℓ, p_ℓ) and (h_r, p_r), respectively, let (H_ℓ, P_ℓ) and (H_r, P_r) be the corresponding Riemann coordinates. Then, the wave size of the jump $\big((h_\ell, p_\ell),\ (h_r, p_r)\big)$ can be defined in two coordinate systems as follows:

- the size of a 1-wave (h-wave) is measured by $\rho_h = H_r - H_\ell$ or $\gamma_h = h_r - h_\ell$ in Riemann or original coordinates, respectively.
- the size of a 2-wave (p-wave) is measured by $\rho_p = P_r - P_\ell$ or $\gamma_p = p_r - p_\ell$ in Riemann or original coordinates, respectively.

Next, at any time $t > 0$ where no interaction occurs and away from time steps, let $\mathcal{J}_i\big(u(t)\big)$ denote a set of indexes α associated to the jumps of the i-th family of $u(t)$ located at x_α and let $p^\ell_\alpha \doteq P(x_\alpha-)$. Also, set $\mathcal{J}\big(u(t)\big) \doteq \mathcal{J}_1\big(u(t)\big) \bigcup \mathcal{J}_2\big(u(t)\big)$ to denote the collection of indexes associated to all jumps of $u(t)$ and $k_\alpha \in \{1,2\}$

the characteristic family of the jump $\alpha \in \mathcal{J}\big(u(t)\big)$, so that, in particular, one has $\alpha \in \mathcal{J}_{k_\alpha}\big(u(t)\big)$. Then, we define the *total strength* of waves in $u(t)$ as:

$$V_i\big(u(t)\big) \doteq \sum_{\alpha \in \mathcal{J}_i((u(t))} |\rho_\alpha|, \qquad i = 1,2, \qquad V\big(u(t)\big) = V_1\big(u(t)\big) + V_2\big(u(t)\big) \doteq \sum_{\alpha \in \mathcal{J}(u(t))} |\rho_\alpha|\,, \tag{2.4}$$

and the *interaction potential* as:

$$\mathcal{Q}\big(u(t)\big) \doteq \mathcal{Q}_{hh} + \mathcal{Q}_{hp} + \mathcal{Q}_{pp}\,. \tag{2.5}$$

where

$$\mathcal{Q}_{hh} \doteq \sum_{\substack{k_\alpha = k_\beta = 1 \\ x_\alpha < x_\beta}} \omega_{\alpha,\beta}|\rho_\alpha||\rho_\beta|\,, \qquad \mathcal{Q}_{hp} \doteq \sum_{\substack{k_\alpha = 2, k_\beta = 1 \\ x_\alpha < x_\beta}} |\rho_\alpha \rho_\beta|, \quad \mathcal{Q}_{pp} \doteq \sum_{(\alpha,\beta) \in Appr_2} |\rho_\alpha \rho_\beta| \tag{2.6}$$

with the weights $\omega_{\alpha,\beta} := \overline{\delta} \cdot \min\{|p^\ell_\alpha - 1|, |p^\ell_\beta - 1|\}$ if ρ_α, ρ_β are 1-shocks lying on the same side of $p = 1$, otherwise $\omega_{\alpha,\beta} := 0$, for a suitable constant $\overline{\delta} > 0$ sufficiently small. Note that $\mathcal{Q}_{hh}$ is the modified interaction potential of waves of the first family (h-waves) introduced in [2] and the others are defined the usual way. Relying on the interaction estimates established in [2], the *Glimm functional*

$$\mathcal{G}\big(u(t)\big) \doteq V\big(u(t)\big) + \mathcal{Q}\big(u(t)\big) \tag{2.7}$$

is nonincreasing in any time interval $]t_k, t_{k+1}[$ between two consecutive time steps. Instead, the estimates derived [2] on the variation of the strength of waves when the solution is updated with the source term, imply that at any time step $t_k = k\Delta t = k\,s$ there holds

$$\mathcal{G}\big(u(t_k+)\big) \le \big(1 + \mathcal{O}(1)\Delta t\big) \cdot \mathcal{G}^-\big(u(t_k-)\big)\,, \tag{2.8}$$

i.e. $\mathcal{G}$ is increasing across t_k.

3 Stability Functional

Let u and $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^n$ be two approximate solutions to (1.1) and consider any piecewice constant function z with the property that for fixed t, $z(t,\cdot) : \mathbb{R} \to \mathbb{R}^2$ is a L^1 function of small total variation. In addition, $z(t,x)$ has finitely many discontinuities that are polygonal lines and the slope of such a line is bounded in absolute value by a fixed number $\hat{\lambda}$. Also, there exists a constant $\sigma > 0$ such that $Tot.Var.z(t) \le \sigma$, for all $t > 0$. We clarify that z is an arbitrary function with the aforementioned properties and is not related to the system (1.1). Next, consider the i-shock curve $\mathbf{S}_i(\cdot;\cdot)$ and the scalar functions η_i $i = 1,2$ defined implicitly by

$$w(t,x) = \mathbf{S}_2(\eta_2(t,x);\cdot) \circ \mathbf{S}_1(\eta_1(t,x); u(t,x))\,, \tag{3.1}$$

where $w \doteq v + z$. According to this definition, the parameter η_i denotes the strength in the original coordinates along the i-shock curves connecting u and $w = v + z$. We clearly have

$$\frac{1}{C_0}|u(x) - w(x)| \le \sum_i |\eta_i(x)| \le C_0|u(x) - w(x)| \tag{3.2}$$

for some constant $C_0 > 0$. We can now define the *stability functional*

$$\Phi_z(u(t), v(t)) \doteq \sum_{i=1}^{2} \int_{-\infty}^{\infty} |\eta_i(x,t)| W_i(x,t)\,dx \tag{3.3}$$

with weights W_i of be

$$W_i(x,t) \doteq 1 + \kappa_1 \mathcal{A}_i(x,t) + \kappa_2\big[\mathcal{G}(u(t)) + \mathcal{G}(v(t))\big], \tag{3.4}$$

for suitable positive constants $\kappa_1 < \kappa_2$ to be specified. Here $\mathcal{G}$ is the Glimm functional defined in (2.4)-(2.7), and $\mathcal{A}_i(t;x)$ measures the total amount of waves in $u(t)$ and $v(t)$ which approach the i-wave η_i located at x defined as follows:

$$\begin{aligned}
\mathcal{A}_1(t;x) \doteq & \sum_{\substack{\alpha \in \mathcal{J}(u) \cup \mathcal{J}(v) \\ k_\alpha = 2,\, x_\alpha < x}} |\rho_\alpha| \\
& + \begin{cases} \Bigg[\displaystyle\sum_{\substack{\alpha \in \mathcal{J}(u), k_\alpha = 1 \\ u_2(x_\alpha -) > 1,\, x_\alpha < x}} + \sum_{\substack{\alpha \in \mathcal{J}(u), k_\alpha = 1 \\ u_2(x_\alpha -) < 1,\, x_\alpha > x}} + \sum_{\substack{\alpha \in \mathcal{J}(v), k_\alpha = 1 \\ v_2(x_\alpha -) > 1,\, x_\alpha > x}} + \sum_{\substack{\alpha \in \mathcal{J}(v), k_\alpha = 1 \\ v_2(x_\alpha -) < 1,\, x_\alpha < x}} \Bigg] |p^\ell_\alpha - 1||\rho_\alpha| & \text{if } \eta_1(t,x) < 0 \\
\Bigg[\displaystyle\sum_{\substack{\alpha \in \mathcal{J}(v), k_\alpha = 1 \\ v_2(x_\alpha -) > 1,\, x_\alpha < x}} + \sum_{\substack{\alpha \in \mathcal{J}(v), k_\alpha = 1 \\ v_2(x_\alpha -) < 1,\, x_\alpha > x}} + \sum_{\substack{\alpha \in \mathcal{J}(u), k_\alpha = 1 \\ u_2(x_\alpha -) > 1,\, x_\alpha > x}} + \sum_{\substack{\alpha \in \mathcal{J}(u), k_\alpha = 1 \\ u_2(x_\alpha -) < 1,\, x_\alpha < x}} \Bigg] |p^\ell_\alpha - 1||\rho_\alpha| & \text{if } \eta_1(t,x) > 0 \end{cases}
\end{aligned} \tag{3.5}$$

and

$$
\mathcal{A}_2(t;x) \doteq \sum_{\substack{\alpha\in\mathcal{J}(u)\cup\mathcal{J}(v)\\ k_\alpha=1,\, x_\alpha>x}} |\rho_\alpha| + \begin{cases} \Bigg[\displaystyle\sum_{\substack{\alpha\in\mathcal{J}(u),k_\alpha=2\\ x_\alpha>x}} + \sum_{\substack{\alpha\in\mathcal{J}(v),k_\alpha=2\\ x_\alpha<x}}\Bigg]|\rho_\alpha| & \text{if } \eta_2(t,x)<0 \\ \Bigg[\displaystyle\sum_{\substack{\alpha\in\mathcal{J}(v),k_\alpha=2\\ x_\alpha>x}} + \sum_{\substack{\alpha\in\mathcal{J}(u),k_\alpha=2\\ x_\alpha<x}}\Bigg]|\rho_\alpha| & \text{if } \eta_2(t,x)>0 \end{cases} \tag{3.6}
$$

where p_α^ℓ denotes the left state of the jump located at x_α and by ρ_α the corresponding strength of the jump in Riemann coordinates.

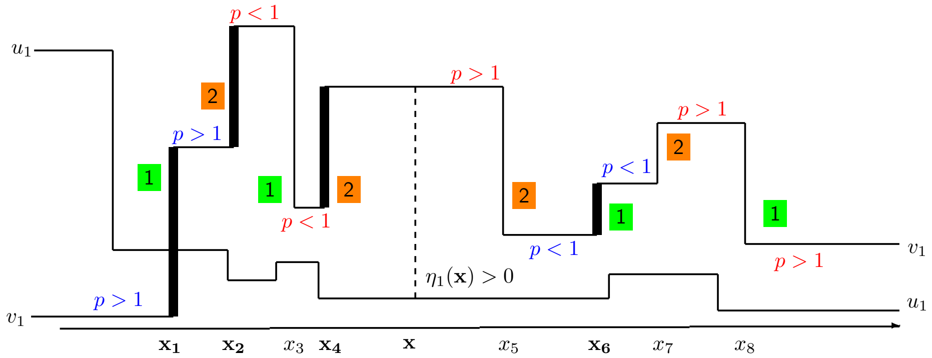

Figure 1: *Approaching waves* in v towards $\eta_1(\mathbf{x})>0$ are indicated by the jumps marked with bolded lines. Also, regions $p<1$, $p>1$ can only be connected by $2-$waves crossing the line $p=1$. The selected $1-$waves that are located at $\mathbf{x}_\alpha$ with $\mathbf{x}_\alpha<\mathbf{x}$ correspond to $\gamma\to\lambda_1(\gamma;\cdot)$ strictly increasing, i.e. $\{p>1\}$. On the other hand, the selected $1-$waves that are located at $\mathbf{x}_\alpha$ with $\mathbf{x}_\alpha>\mathbf{x}$ correspond to $\gamma\to\lambda_1(\gamma;\cdot)$ strictly decreasing, i.e. $\{p<1\}$.

Notice that the main novelty of our functional is encoded in the weight W_1 and in particular in $\mathcal{A}_1$, whereas W_2 has almost the same expression of the weight given in [9] for GNL and LD characteristic fields. In fact, the only difference between the definition of the weight W_2 here and the one given in [9] relies in the presence of the whole Glimm functional $\mathcal{G}$ of u and v in W_i, instead of their interaction potential $\mathcal{Q}$. Indeed, in comparison to the weights W_i used in [6, § 8], here the terms of the *Glimm functionals* $\mathcal{G}$ and not only the *interaction potential* $\mathcal{Q}$ are needed in the definition of W_i to control the change $\mathcal{A}_i$ across an interaction time. This is due to the fact that, since the first characteristic family is not GNL, we adopt as in [2] a wave interaction potential $\mathcal{Q}$, suited to (1.1), that is in general not decreasing in presence of interactions of 1-waves of different sign (1-shocks with 1-rarefaction waves). Therefore, one needs to exploit the decrease of the total strength V of waves due to cancellation in order to control the possible increase of the potential interaction $\mathcal{Q}$ occurring at such interactions.

Instead, because of the properties of the non GNL first characteristic family, the definition of 1-waves approaching η_1 varies if the left state of such waves lies on the left or on the right of $\{p=1\}$ (see Figure 1). The key ingredient in the definition of $\mathcal{A}_1$ is the appropriate formulation of *approaching wave* of the first family for a given wave $\eta_1(x)$ in the jump $(u(x),v(x))$, which extends to our case the definition given in [9] for GNL characteristic fields. Observe that, letting $\gamma\mapsto\mathbf{S}_1(\gamma;\,h_0,p_0)$ be the Rankine-Hugoniot curve of right states of the first family issuing from a given state $(h_0,p_0)\in\Omega$, and denoting $\lambda_1(\gamma;\,h_0,p_0)$ the Rankine-Hugoniot speed of the jump connecting (h_0,p_0) with $\mathbf{S}_1(\gamma;\,h_0,p_0)$, by the properties of system (1.1) it follows that $\gamma\mapsto\lambda_1(\gamma;\,h_0,p_0)$ is strictly increasing on $\{p>1\}$, strictly decreasing on $\{0<p<1\}$, and constant along $\{p=1\}$. Therefore, if the size $\eta_1(x)$ is positive, we shall regard as approaching all the 1-waves present in v which either have left state in the region $\{p>1\}$ and are located on the left of $\eta_1(x)$, or have left state in the region $\{0<p<1\}$ and are located on the right of $\eta_1(x)$. On the contrary, we regard as approaching to $\eta_1(x)>0$ all the 1-waves present in u which either have left state in the region $\{p>1\}$ and are located on the right of $\eta_1(x)$, or have left state in the region $\{0<p<1\}$ and are located on the left of $\eta_1(x)$. Similar definition is given in the case where $\eta_1(x)<0$.

Moreover, in [9], the weights W_i are expressed only in terms of the strength of the approaching waves. Instead here the terms of $\mathcal{A}_1$ related to the approaching waves of the first family have the form of the product of the strength of the waves $|\rho_\alpha|$ times the distance from $\{p=1\}$ of the left state of the waves $|p_\alpha-1|$. The presence of the factor $|p_\alpha-1|$ is crucial to guarantee the decreasing property of $\Phi_z(u(t,\cdot),v(t,\cdot))$ at times of interactions involving a 1-wave, say of strength $|\rho_\alpha|$, and a 2-wave crossing $\{p=1\}$ (i.e. connecting two states lying on opposite sides of $\{p=1\}$), say of strength $|\rho_\beta|$. In fact, in this case the possible increase of $\mathcal{A}_1$ turns out to be of order $|p_\beta-1||\rho_\alpha|\approx|\rho_\alpha\rho_\beta|$, and thus it can be controlled by the decrease of $\mathcal{G}$ determined by the corresponding decrease of the interaction potential. Unfortunately, because of the presence of these quadratic terms in the weight W_1, we are forced to establish sharp fourth order interaction estimates in order to carry on the analysis of the variation of $\Phi_z(u(t,\cdot),v(t,\cdot))$. This is achieved deriving accurate Taylor expansions of the Hugoniot and rarefaction curves of each famiy, which rely on the specific geometric features of system (1.1). Namely, the rarefaction and Hugoniot curves through the same point are "almost" straight lines and have "almost" third order tangency at their issuing point near $\{p = 1\}$ for the first family and near $\{h = 0\}$ for the second family. We say that the characteristic fields of (1.1) are "almost Temple class".

4 Main Theorems

It should be noted that for fixed κ_1 and κ_2, the functional W_i is locally bounded. Hence, the functional Φ_z is equivalent to the $\mathbf{L^1}$ distance between $u(t)$ and $w(t) = v(t) + z(t)$:

$$\frac{1}{C_0}\big\|u(t) - w(t)\big\|_{L^1} \le \Phi_z(u(t), v(t)) \le C_0 \cdot W^* \cdot \big\|u(t) - w(t)\big\|_{L^1} \qquad \forall\, t > 0\,. \tag{4.1}$$

In the same spirit of [9], we prove that Φ_z is "almost decreasing" in time if the only effect of the convective part of (1.1), otherwise it is exponentially increasing in time with the increase to be estimated using the operator splitting scheme. To prove this, we clarify that the functional $\Phi_z(u, v)$ in (3.3) is employed in two ways: either when both u and v are approximate solutions to the non-homogeneous system (1.1) and $z \equiv 0$ or when u and v are approximate solutions to the homogeneous system (2.2) and $z \neq 0$ is arbitrary.

First, consider domains $\mathcal{D}$ of the form

$$\begin{aligned}\mathcal{D}(M_0, p_0, \delta_0) = cl\big\{(h, p) \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}^2) :\ & h, p \ \text{ are piecewise constant,}\\ & 0 \le h(x) \le \delta_0,\ p(x) \ge p_0 \ \text{ for a.e. } x,\\ & \text{and } \ \mathrm{TotVar}\{(h, p)\} \le M_0, \quad \|h\|_{\mathbf{L}^1} + \|p - 1\|_{\mathbf{L}^1} \le M_0\big\},\end{aligned} \tag{4.2}$$

where cl denotes the $\mathbf{L}^1$-closure, $\mathrm{TotVar}\{(h, p)\} \doteq \mathrm{TotVar}\{h\} + \mathrm{TotVar}\{p\}$, and M_0, p_0, δ_0 are positive constants. Given $M_0, p_0 > 0$, we prove in [3] that there exist constants $\delta_0, \delta_0^*, p_0^*, p_1^*, \kappa_1, \kappa_2, \sigma, C_1, C_2 > 0$, so that, letting Φ_z be the functional defined in (3.3)-(3.6), the followings hold true.

(i) Let u and $v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2$ be two *ε-front tracking approximate solution* to (2.2) with initial data $u(\,\cdot, 0), v(\,\cdot, 0) \in \mathcal{D}(M_0, p_0, \delta_0)$ and with values in $[0, \delta_0^*] \times [p_0^*, p_1^*]$. Let z be a piecewise constant function as in Section 3, then

$$\Phi_z\big(u(\tau_2), v(\tau_2)\big) \le \Phi_z\big(u(\tau_1), v(\tau_1)\big) + C_1 \cdot \big(\varepsilon + \sigma\big)(\tau_2 - \tau_1) \qquad \forall\, \tau_2 > \tau_1 > 0\,. \tag{4.3}$$

(ii) Let u and $v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2$ be two s-ε-approximate solution of (1.1) with initial data $u(\,\cdot, 0), v(\,\cdot, 0) \in \mathcal{D}(M_0, p_0, \delta_0)$ and with values in $[0, \delta_0^*] \times [p_0^*, p_1^*]$. Then, letting $t_k \doteq k\Delta t = k\, s$, $\ (k \in \mathbb{N})$ be the time steps, there holds

$$\Phi_0\big(u(\tau_2), v(\tau_2)\big) \le \Phi_0\big(u(\tau_1), v(\tau_1)\big) + C_1 \cdot \varepsilon(\tau_2 - \tau_1) \qquad \forall\, t_k < \tau_1 < \tau_2 < t_{k+1}\,, \tag{4.4}$$

and

$$\begin{aligned}\Phi_0\big(u(t_k+), v(t_k+)\big) \le\ & \Phi_0\big(u(t_h+), v(t_h+)\big)\big(1 + C_2 \cdot \Delta t\big)^{(k-h)} +\\ & + C_1 \cdot \varepsilon\, \Delta t \sum_{i=1}^{k-h} \big(1 + C_2 \cdot \Delta t\big)^{i} \qquad \forall\, 0 \le h < k,\end{aligned} \tag{4.5}$$

for all $k \in \mathbb{N}$.

The proofs of (i) and (ii) above can be found in [3, §4]. By estimate (4.3), the front tracking approximate solutions to the homogeneous system (2.2) converge to a unique limit, depending Lipschitz continuously on the initial data in the L^1 norm, that defines a semigroup solution operator $\mathcal{S}_t$, $t \ge 0$, on the domains $\mathcal{D}$ defined above. In other words, for any given initial data $\overline{u} \doteq (\,\overline{h}, \overline{p}\,) \in \mathcal{D}(M_0, p_0, \delta_0)$, the map $u(t, x) \doteq \mathcal{S}_t \overline{u}(x)$ provides an entropy weak solution of the Cauchy problem for (2.2)–(2.3). The statement is the following:

Theorem 4.1. *Given $M_0, p_0 > 0$, there exist $\delta_0, \delta_0^*, M_0^*, p_0^*, L > 0$ and a unique (up to the domain) semigroup map*

$$\mathcal{S} : [0, +\infty) \times \mathcal{D}_0 \to \mathcal{D}_0^*, \qquad\qquad (\tau, \overline{u}) \mapsto \mathcal{S}_\tau \overline{u}\,, \tag{4.6}$$

with $\mathcal{D}_0 \doteq \mathcal{D}(M_0, p_0, \delta_0)$, $\mathcal{D}_0^* \doteq \mathcal{D}(M_0^*, p_0^*, \delta_0^*)$ domains defined as in (4.2), which enjoys the following properties:

(i) $\mathcal{S}_{\tau_2}\big(\mathcal{S}_{\tau_1}\, \overline{u}\big) \in \mathcal{D}_0^* \qquad \forall\, \overline{u} \in \mathcal{D}_0, \ \ \forall\, \tau_1, \tau_2 \ge 0;$

(ii) $\mathcal{S}_0\, \overline{u} = \overline{u}, \quad \mathcal{S}_{\tau_1 + \tau_2}\, u = \mathcal{S}_{\tau_2}\big(\mathcal{S}_{\tau_1}\, \overline{u}\big) \qquad \forall\, \overline{u} \in \mathcal{D}_0, \ \ \forall\, \tau_1, \tau_2 \ge 0;$

(iii) $\big\|\mathcal{S}_{\tau_2}\overline{u} - \mathcal{S}_{\tau_1}\overline{v}\big\|_{\mathbf{L}^1} \le L \cdot \big(|\tau_1 - \tau_2| + \|\overline{u} - \overline{v}\|_{\mathbf{L}^1}\big) \qquad \forall\, \overline{u}, \overline{v} \in \mathcal{D}_0, \ \ \forall\, \tau_1, \tau_2 \ge 0;$

(iv) For any $\overline{u} \doteq (\overline{h}, \overline{p}) \in \mathcal{D}_0$, the map $\big(h(x,\tau), p(x,\tau)\big) \doteq \mathcal{S}_\tau\, \overline{u}(x)$ provides an entropy weak solution of the Cauchy problem (2.2), (2.3). *Moreover, $\mathcal{S}_\tau\, \overline{u}(x)$ coincides with the unique limit of front tracking approximations.*

(v) If $\overline{u} \in \mathcal{D}_0$ is piecewise constant, then for τ sufficiently small $u(\,\cdot\,,\tau) \doteq \mathcal{S}_\tau\, \overline{u}$ coincides wtth the solution of the Cauchy problem (2.2), (2.3) *obtained by piecing together the entropy solutions of the Riemann problems determined by the jumps of $\overline{u}$.*

It should be noted that the image of the map $\mathcal{S}_t$ in (4.8) is the same for every $t > 0$, but the domain $\mathcal{D}_0$ is not positively invariant under the action of $\mathcal{S}$. Indeed, it turns out that the $\mathbf{L}^\infty$, $\mathbf{L}^1$- norms as well as the total variation of the solution (that appear in the definition of the domain (4.2)) may well increase in presence of interactions (see the analysis in [2, Section 5]).

Moreover, relying on (4.4)–(4.5) and on Theorem 4.1, we prove that approximate solutions of (1.1) generated by a front-tracking algorithm combined with an operator splitting scheme, in turn, converge to a map that defines a Lipschitz continuous semigroup operator $\mathcal{P}_t$, $t \geq 0$, on domains as (4.2), with a Lipschitz constant that grows exponentially in time and the trajectories $u(t) = \mathcal{P}_t \bar{u}$ are entropy weak solution of the Cauchy problem (1.1), (2.3). Let us point out that, although the source term of system (1.1) is not dissipative, relying on the global existence result established in [2], we construct a semigroup map whose image $\mathcal{D}_0^*$ is the same for every time $t > 0$. Also, the uniqueness of the limit of approximate solutions to (1.1) and of the semigroup operator $\mathcal{P}$, is achieved as in [1] deriving the key estimate

$$\left\|\mathcal{P}_\theta \overline{u} - \mathcal{S}_\theta \overline{u} - \theta \cdot \big((\overline{p}-1)\overline{h}\,\big)\right\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\theta^2 \qquad \text{as} \qquad \theta \to 0\,, \tag{4.7}$$

relating the solutions operators of the homogeneous and nonhomogeneous systems, and invoking a general uniqueness result for quasidifferential equations in metric spaces [5]. Here is our theorem:

Theorem 4.2. *Given $M_0, p_0 > 0$, there exist $\delta_0, \delta_0^*, M_0^*, p_0^*, L', C > 0$ so that the conclusions of Theorem 4.1 hold together with the following. There exist a map*

$$\mathcal{P} : [0, +\infty) \times \mathcal{D}_0 \to \mathcal{D}_0^*, \qquad\qquad (\tau, \overline{u}) \mapsto \mathcal{P}_\tau \overline{u}\,, \tag{4.8}$$

(with $\mathcal{D}_0, \mathcal{D}_0^*$ domains as in (4.2)*), which enjoys the properties:*

(i) $\mathcal{P}_{\tau_1}\big(\mathcal{P}_{\tau_2}\,\overline{u}\big) \in \mathcal{D}_0^* \qquad \forall\, \overline{u} \in \mathcal{D}_0, \ \ \forall\, \tau_1, \tau_2 \geq 0;$

(ii) $\mathcal{P}_0 \overline{u} = \overline{u}, \quad \mathcal{P}_{\tau_1+\tau_2} u = \mathcal{P}_{\tau_2}\big(\mathcal{P}_{\tau_1}\overline{u}\big) \qquad \forall\, \overline{u} \in \mathcal{D}_0, \ \ \forall \tau_1, \tau_2 \geq 0;$

(iii) $\left\|\mathcal{P}_{\tau_1}\overline{u} - \mathcal{P}_{\tau_2}\overline{v}\right\|_{\mathbf{L}^1} \leq L'\big(e^{C_4\cdot\tau_2}\cdot\|\overline{u}-\overline{v}\|_{\mathbf{L}^1} + (\tau_2-\tau_1)\big) \qquad \forall\, \overline{u},\, \overline{v} \in \mathcal{D}_0, \ \ \forall\, \tau_2 > \tau_1 > 0\,,$

(iv) For any $\overline{u} \doteq (\overline{h}, \overline{p}) \in \mathcal{D}_0$, the map $\big(h(x,\tau), p(x,\tau)\big) \doteq \mathcal{P}_\tau \overline{u}(x)$ provides an entropy weak solution of the Cauchy problem (1.1), (2.3).

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References

[1] D. Amadori and G. Guerra, Uniqueness and continuous dependence for systems of balance laws with dissipation, *Nonlinear Anal.* **49(7)** (2002) 987–1014.

[2] D. Amadori and W. Shen, Global existence of large BV solutions in a model of granular flow, *Comm. Part. Diff. Equations* **34** (2009) 1003–1040.

[3] F. Ancona, L. Caravenna and C. Christoforou, Exponential stability of large BV solutions in a model of granular flow, Preprint (2019).

[4] F. Ancona, A. Marson, Well-posedness for general 2×2 systems of conservation laws, *Mem. Amer. Math. Soc.*, **169** (2004) (801).

[5] A. Bressan, On the Cauchy problem for systems of conservation laws, Actes du 29ème Congrès d'Analyse Numérique: CANum'97 (Larnas, 1997) Soc. Math. Appl. Indust., Paris, 1998, *ESAIM Proc.*, 3, 23–36 (electronic).

[6] A. Bressan, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*. Oxford Lecture Series in Mathematics and its Applications, 20. Oxford University Press, 2000.

[7] A. Bressan, R.M. Colombo, The semigroup generated by 2×2 systems of conservation laws, *Arch. rational Mech. Anal.*, **133** (1996), pp. 1-75.

[8] A. Bressan, G. Crasta, B. Piccoli, Well-posedness of the Cauchy problem for $n \times n$ conservation laws, *Amer. Math. Soc. Memoir*, **146** (2000) (694).

[9] A. Bressan, T.P. Liu, T. Yang, L^1 stability estimates for $n \times n$ conservation laws, *Arch. Rational Mech. Anal.*, **149**, 1-22 (1999).

[10] C. Christoforou, Hyperbolic systems of balance laws via vanishing viscosity, *J. Differential Equations* **221** (2006), 470–541.

[11] C. M. Dafermos and L. Hsiao, Hyperbolic systems of balance laws with inhomogeneity and dissipation. *Indiana U. Math. J.* **31** (1982) 471– 491.

[12] K. P., Hadeler, and C. Kuttler, Dynamical models for granular matter. *Granular Matter* **2** (1999) 9–18.

[13] T.P. Liu, T. Yang, L^1-stability for 2×2 systems of hyperbolic conservation laws, *J. Amer. Math. Soc.*, **12** (1999), no. 3, pp. 729-774.

[14] W. Shen, On the shape of avalanches. *J. Math. Anal. Appl.* **339** (2008) 828–838.