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A counter-example concerning regularity properties for systems of conservation laws

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We exhibit an explicit counter-example which rules the possibility of extending to systems of conservation laws a regularity property of scalar conservation laws known as Schaeffer's Theorem. Loosely speaking, Schaeffer's Regularity Theorem asserts that, for a generic smooth initial datum, the solution of the Cauchy problem can develop at most finitely many discontinuities on compact sets.

Systems of conservation laws are a class of partial differential equations with several applications coming from both physics and engineering, in particular from the fluid-dynamics: we refer to the book by Dafermos [5] for an extensive introduction. In the case when the space variable is one dimensional, systems of conservation laws take the form

$$\partial_t U + \partial_x \big[F(U)\big] = 0. \tag{1}$$

In the previous expression, the unknown U depends on $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and attains values in $\mathbb{R}^n$. The so-called flux function $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and depends on the underlying physical model. In the following, we always impose a standard hypothesis of *strict hyperbolicity*, namely we require that the Jacobian matrix $DF(U)$ admits n real and distinct eigenvalues. This hypothesis is satisfied in several physically relevant examples, like the Euler equations of fluid-dynamics. We focus on the Cauchy problem, which is posed by coupling (1) with the initial datum

$$U(0, x) = U_0(x), \qquad x \in \mathbb{R}. \tag{2}$$

From the mathematical viewpoint, the study of systems of conservation laws poses two main challenges. First, the finite time breakdown of classical solutions. Even if the initial datum U_0 is smooth, in general the classical solution of the Cauchy problem (1)-(2) is only defined on a finite time interval, owing to the formation of so-called *shocks*, namely discontinuities that propagate in time. Note that examples of finite time breakdown of classical solutions are available even for the simplest possible nonlinear conservation laws, namely the Burgers'equation, which corresponds to the choices $n = 1$ and $F(U) = U^2/2$. Since classical solutions do not exist globally in time, it is natural to introduce a notion of *distributional solution*.

The second main challenge is the non uniqueness of distributional solutions: in general, the Cauchy problem (1)-(2) admits *infinitely many* distributional solutions. In the attempt at restoring uniqueness, various *admissibility criteria* have been introduced: they are additional conditions imposed on the solution and they are often motivated by physical considerations. As a matter of fact, in all the cases we consider in the following the *admissible solution* of the Cauchy problem (1)-(2) can be recovered as the limit of the second order approximation, see again [5] for an extended discussion on admissibility criteria.

In the following, we are concerned with *regularity properties* of the admissible solution. More precisely, we stress once more that in general the admissible solution of the Cauchy problem (1)-(2) does develop shocks and we aim at addressing the following question.

Question 1: assume that the initial datum U_0 is smooth. Can we conclude that the number of shocks of the admissible solution of the Cauchy problem (1)-(2) is locally finite, namely it is finite on every compact subset of the (t, x) plane?

Before entering the technical details, we provide some motivation by pointing out that knowing that the admissible solution has at most finitely many shocks on compact sets could considerably simplify the study of several approximation schemes. In particular, it would help in the design of efficient numerical schemes: we refer to the book by LeVeque [9] for a more extended discussion.

In general, the answer to **Question 1** is negative: indeed, even in the case of the Burgers' equation various authors have provided examples of smooth initial data such that the admissible solutions of the corresponding Cauchy problem develop infinitely many shocks on a compact set. Notwithstanding the above obstruction, the Schaeffer Regularity Theorem [7] states that for a *generic* initial datum the answer to **Question 1** is positive, provided that equation (1) is scalar (i.e., the unknown U is real-valued, $n = 1$) and the flux function F is uniformly convex, namely $F'' \geq c$ for some constant $c > 0$. The term *generic* is here to be interpreted in a suitable technical sense, which is specified below. Note that in the following statement we denote by $\mathcal{S}(\mathbb{R})$ the Schwarz space of rapidly decreasing functions, see [6, p.133] for the precise definition.

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Theorem 1 (Schaeffer, 1973) *Assume that $n = 1$ and that the flux F is smooth and uniformly convex, namely $F''(U) \geq c > 0$ for some constant $c > 0$ and for every $U \in \mathbb{R}$. Then there is a set $\mathfrak{F} \subseteq \mathcal{S}(\mathbb{R})$ that enjoys the following properties:*

1. *$\mathfrak{F}$ is open and dense in $\mathcal{S}(\mathbb{R})$.*
2. *For every $U_0 \in \mathfrak{F}$, the number of shocks of the entropy admissible solution of the Cauchy problem* (1)-(2) *is locally finite.*

See [8] for an explicit characterization of the set $\mathfrak{F}$. We stress again that the proof of Schaeffer's Theorem only works in the scalar case, namely when the unknown U attains values in $\mathbb{R}$. We now want to address the following question.

Question 2: can we extend Schaeffer's Theorem to the case of systems, namely to the case when the unknown function U in (1) attains values in $\mathbb{R}^n$?

Before providing an answer to the above question, we have to introduce some further notation. We recall that *strict hyperbolicity* dictates that the Jacobian matrix $DF(U)$ admits n real and distinct eigenvalues $\lambda_1(U), \dots, \lambda_n(U)$. For every $i = 1, \dots, n$ the i-th vector field is termed *genuinely non linear* if

$$\nabla \lambda_i(U) \cdot \vec{r}_i(U) \geq c > 0, \quad \text{for every } U \in \mathbb{R}^n \tag{3}$$

and for some suitable constant $c > 0$. Note that Schaeffer's Theorem holds under the assumption that the flux function F in (1) is uniformly convex. This condition only makes sense in the scalar case, i.e. when $n = 1$. As a matter of fact, the analogous in the case of systems $n > 1$ of the condition that F is convex is the requirement that every characteristic field is genuinely nonlinear, namely (3) holds for $i = 1, \dots, n$. Indeed, some relevant regularity results that are known to hold for *scalar* conservation laws with convex fluxes can be extended to *systems* of conservation laws where every characteristic field is genuinely nonlinear, see [5] for a more extended discussion and [2] for a recent related example. For this reason, a more correct formulation of **Question 2** could be the following: can we extend Schaeffer's Theorem to systems where every characteristic field is genuinely nonlinear? The answer to the above question is negative: the following result provides an explicit counter-example.

Theorem 2 *There are a flux function $F : \mathbb{R}^3 \to \mathbb{R}^3$, a compact set $K \subseteq \mathbb{R}^+ \times \mathbb{R}$ and a set $\mathfrak{B} \subseteq \mathcal{S}(\mathbb{R})$ that enjoy the following properties:*

i. *system* (1) *is strictly hyperbolic and every characteristic field is genuinely nonlinear.*

ii. *The set $\mathfrak{B}$ is non empty and open in $\mathcal{S}(\mathbb{R})$.*

iii. *For every $U_0 \in \mathfrak{B}$ the admissible solution of the Cauchy problem* (1)-(2) *has infinitely many shocks in K.*

Note that condition iii. implies that, for every $U_0 \in \mathfrak{B}$, the number of shocks developed by the admissible solution cannot be locally finite. Owing to condition ii., this implies that the set of initial data such that the number of shocks of the admissible solution is locally finite cannot be dense in $\mathcal{S}(\mathbb{R})$. By recalling property 1. in the statement of Shaeffer's Theorem, we infer that the above result provides a counter-example to the extension of Shaeffer's Theorem to the case of systems.

The proof of Theorem 2 is given in [4], see also [3] for a related counter-example. An interesting feature of the proof is that the construction of the counter-example is completely explicit, in particular we can provide a formula for both the flux function F and the sets $\mathfrak{B}$ and K.

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