NEW INTERACTION ESTIMATES FOR THE BAITI-JENSSEN SYSTEM

Laura Caravenna
Dipartimento di Matematica,
Università degli Studi di Padova,
via Trieste 63, 35121 Padova, Italy

Laura V. Spinolo
IMATI-CNR,
via Ferrata 1, 27100 Pavia, Italy

Abstract. We establish new interaction estimates for a system introduced by Baiti and Jenssen. These estimates are pivotal to the analysis of the wave front-tracking approximation. In a companion paper we use them to construct a counter-example which shows that Schaeffer's Regularity Theorem for scalar conservation laws does not extend to systems. The counter-example we construct shows, furthermore, that a wave-pattern containing infinitely many shocks can be robust with respect to perturbations of the initial data. The proof of the interaction estimates is based on the explicit computation of the wave fan curves and on a perturbation argument.

1. Introduction

We deal with the system of conservation laws

$$\partial_t U + \partial_x \big[F_\eta(U)\big] = 0. \tag{1}$$

The unknown $U = U(t,x)$ attains values in $\mathbb{R}^3$:

$$\begin{array}{cccc} U: & [0,+\infty[\times\mathbb{R} & \to & \mathbb{R}^3 \\ & (t,x) & \mapsto & U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \end{array}$$

and the flux function $F_\eta : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$F_\eta(U) := \begin{pmatrix} 4\big[(v-1)u - w\big] + \eta p_1(U) \\ v^2 \\ 4\Big\{v(v-2)u - (v-1)w\Big\} + \eta p_3(U) \end{pmatrix}. \tag{2}$$

In the previous expression, the parameter η attains values in the interval $[0, 1/4[$ and to simplify the exposition we fix the functions p_1 and p_3 by setting

$$p_1(U) = 2uw - 2u^2(v-1), \tag{3}$$

$$p_3(U) = w^2 - u^2(v-2)v. \tag{4}$$

1991 *Mathematics Subject Classification.* Primary: 35L65.

Key words and phrases. conservation laws, regularity, shock formation, Schaeffer Theorem, counter-example.

Note, however, that for some of the results discussed in the following the precise expression of the functions p_1 and p_3 is irrelevant.

System (1),(2) was introduced by Baiti and Jenssen in [3, 19] and it was used to construct an example of a Cauchy problem where the initial data have finite, but large, total variation and the L^∞-norm of the admissible solution blows up in finite time. More recently, the authors of the present paper used the Baiti-Jenssen system (1) to exhibit an explicit counter-example which shows that Schaeffer's regularity result for scalar conservation laws does not extend to systems, see [11]. The counter-example we construct shows, furthermore, that a wave-pattern containing infinitely many shocks can be robust with respect to perturbations of the initial data. We refer to § 2.1 in the present paper for a brief overview of these counter-examples. See also [10].

This note aims at establishing new quantitative interaction estimates for the Baiti-Jenssen systems (1),(2). The estimates we obtain are pivotal to the analysis of the so-called wave front-tracking approximation of the Cauchy problem obtained by coupling (1) with an initial datum $U(0, \cdot) = U_0$. We refer to [5, 14, 18] for an extended discussion on the wave front-tracking approximation. Here we only mention that the wave front-tracking algorithm is based on the construction of a piecewise constant approximation of the Cauchy problem. Under suitable conditions on the initial datum U_0 and on the flux function F_η, one can show that the wave front-tracking approximation converges to an *admissible solution* of the Cauchy problem, see in particular the analysis in [5]. In [11] we construct wave front-tracking approximations of the Cauchy problems obtained by coupling (1) with suitable initial data. We then rely on the wave front-tracking approximation to establish qualitative properties of the limit solutions. In the following we do not consider all the possible interactions one has to handle when constructing the wave front-tracking approximation. We only discuss those that we encounter in [11] and that cannot be handled by relying on straightforward considerations on the structure of the flux F_η.

Before going into the technical details, we make some further remarks. First, in the present note we fix a very specific system in the wider class considered in [3]. The motivation for this choice is twofold: i) it simplifies the notation and ii) the analysis in the present note is sufficient for the applications in [11]. Note, moreover, that in the proof of Lemma 1.1 we use (although not in an essential way) the exact expression of the function F_η evaluated at $\eta = 0$. However, we are confident that our results can be extended to wider classes of systems of the type considered in [3].

Second, in this note the only occurrences where we explicitly use the precise expression of the functions p_1 and p_3 is in the results discussed in § 2.3. More precisely, the proofs of Lemmas 1.1 and 1.2 both rely on a perturbation argument: we first show that our system verifies the statement of the lemmas in the case $\eta = 0$ and then we show that the same holds provided η is sufficiently small. The proof of Lemma 1.2 is completely independent of the specific expression of p_1 and p_3. In the perturbation argument in the proof of Lemma 1.1 we use some results from § 2.3, but we never directly use the specific expression of p_1 and p_2.

Third, the Baiti-Jenssen (1) system in not physical, in the sense that it does not admit strictly convex entropies, see [3] for a proof. It is natural to wonder whether or not the results established in the present note can be extended to physical systems. Very loosely speaking, by combining Lemmas 1.2 and 1.1 below with the analysis in [11, §3.1-3.2] we get the following statement. Under suitable conditions, the only waves generated at the interactions between two shocks are shock waves, or, more precisely, no rarefaction waves are generated at the interaction between two shocks. There are actually several physical systems that share this property: for instance, one can consider the 2×2 example discussed by DiPerna in [15, §5] and assume that the data have sufficiently small total variation. We refer to [6, §4] for the analysis of shock interactions for this system. On the other hand, a much more challenging question is whether or not there is any physical system that exhibit the same behaviors as those discussed in [3, 11]. In other words, one can wonder whether or not a physical system can i) exhibit finite time blow up or ii) violate the regularity prescribed, for scalar conservation laws, by Schaeffer's Theorem. To the best of the authors' knowledge, the answers to the above questions is presently open.

We now give some technical details about the estimates we establish. First, we point out that the Baiti-Jenssen system (1) is strictly hyperbolic in the unit ball, which amounts to say that the Jacobian matrix DF_η admits three real and distinct eigenvalues

$$\lambda_1(U) < \lambda_2(U) < \lambda_3(U) \tag{5}$$

for every U such that $|U| < 1$. Also, if $\eta > 0$ every characteristic field is genuinely nonlinear. In other words, let $\vec{r}_1, \dots, \vec{r}_3$ denote the right smooth eigenvectors associated to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then

$$\nabla \lambda_i(U) \cdot \vec{r}_i(U) \geq c > 0 \tag{6}$$

for some suitable constant $c > 0$ and for every $i = 1, 2, 3$ and $|U| < 1$. In the following, we distinguish three families of shocks: we term a given shock 1-, 2- or 3-shock depending on whether the speed of the shock is close to λ_1, λ_2 or λ_3.

We also point out that establishing interaction estimates for system (1) boils down to the following. Consider the so-called Riemann problem, namely the Cauchy problem obtained by coupling (1) with an initial datum in the form

$$U(0, x) := \begin{cases} U_\ell & x < 0 \\ U_r & x > 0, \end{cases} \tag{7}$$

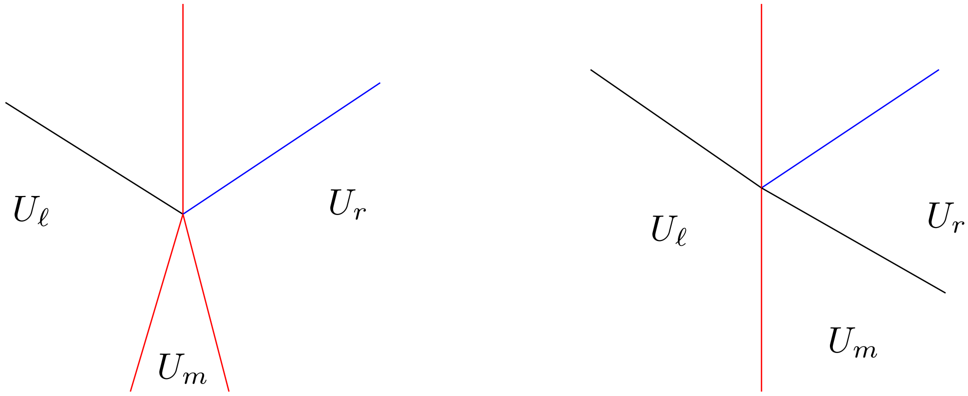
where U_ℓ, $U_r \in \mathbb{R}^3$ are constant states. The above problem admits, in general, infinitely many distributional solutions: we term *admissible* the solution constructed by Lax in the pioneering work [21], see § 2.2 for a brief overview. Establishing interaction estimates for (1) amounts to establish estimates on the admissible solution of the Riemann problem (1)-(7) in the case when U_ℓ and U_r satisfy suitable structural assumptions.

The first case we consider is the case of the interaction of two 2-shocks, see Figure 1, left part. In other words, we assume that there is a state $U_m \in \mathbb{R}^3$ such that

- U_ℓ and U_m are the left and the right states of a Lax admissible 2-shock,
- U_m and U_r are the left and the right states of a Lax admissible 2-shock and
- the shock between U_ℓ and U_m has higher speed than the shock between U_m and U_r.

We now give an heuristic formulation of our interaction estimate and we refer to § 3 for the rigorous statement, which requires some technical notation. Here we only point out that the *strength* of a shock is a quantity defined in § 2.2 which is proportional to the modulus of the difference between the left and the right state of the shock.

Figure 1. Left: interaction between two 2-shocks. Right: interaction between a 2-shock and a 1-shock.



Lemma 1.1. *Fix a constant a such that* $0 < a < 1/2$ *and set* $U_\sharp := (a, 0, -a)$. *Consider the interaction between two 2-shocks and assume that the states* U_ℓ *and* U_r *are sufficiently close to* $U_\sharp$. *If the strengths of the interacting 2-shocks are sufficiently small, then the admissible solution of the Riemann problem* (1)-(7) *is obtained by patching together a 1-shock, a 2-shock and a 3-shock.*

We remark that the relevant point in the above result is that the solution of the Riemann problem that we consider in the statement contains no rarefaction wave.

The second case we consider is the case of the interaction between a 1-shock and a 2-shock, see Figure 1, right part. In other words, we assume that there is a state $U_m \in \mathbb{R}^3$ such that

- U_ℓ and U_m are the left and the right states of a Lax admissible 2-shock,
- U_m and U_r are the left and the right states of a Lax admissible 1-shock.

The case of the interaction of a 3-shock with a 2-shock is analogous. We now give an heuristic formulation of our result and we refer to § 4 for the rigorous statement.

Lemma 1.2. *Consider the interaction between a* 1*-shock and a* 2*-shock and assume both shocks have sufficiently small strength. Then the admissible solution of the Riemann problem* (1)-(7) *is obtained by patching together a 1-shock, a 2-shock and a 3-shock. Also, we establish quantitative bounds from above and from below on the strength of the outgoing shocks, see formulas* (35).

Note that the fact that the three outgoing waves are shocks follows from the analysis in [3]. Also, the bound from above on the strength of the outgoing 3-shocks follows from by now classical interaction estimates, see [5, Page 133, (7.31)]: the main novelty in Lemma 1.2 is that we have a new bound from below on the strength of the outgoing 3-shock, see the left hand side of formula (35). This estimate is important for the analysis in [11].

This note is organized as follows. In § 2 we go over some previous results. In particular, in § 2.1 we provide some motivation for studying the Baiti-Jenssen system (1) by describing two counter-examples that use it. In § 2.2 we recall some results from [21] and in § 2.3 we apply these results to the Baiti-Jenssen system. In § 3 we discuss the interaction of two 2-shocks and in § 4 the interaction of a 1-shock and a 2-shock.

2. Overview of previous results

For the reader's convenience, in this section we go over some previous results. More precisely:

- § 2.1: we discuss two counter-examples based on the Baiti-Jenssen system (1): the original one in [3] and a more recent one devised in [11].
- § 2.2: we follow the famous work by Lax [21] and we outline the construction of the solution of the Riemann problem.
- § 2.3: we apply Lax's construction to the Baiti-Jenssen system.

2.1. Counter-examples based on the Baiti-Jenssen system.

This paragraph is organized as follows:

- § 2.1.1: we discuss the counter-example in [3]
- § 2.1.2: we discuss the counter-example in [11].

Before dealing with the specific examples, we recall two main features of the Baiti-Jenssen system: first, it is strictly hyperbolic, namely (5) holds. Note that strict hyperbolicity is a standard hypothesis for results concerning systems of conservation laws, see [14]. Also, if $\eta > 0$ every characteristic field is genuinely nonlinear, which means that (6) is satisfied for every $i = 1, 2, 3$. This is a remarkable property because loosely speaking systems where all the characteristic field are genuinely nonlinear are usually better behaved than general systems. For instance, the celebrated decay estimate by Oleĭnik [23], which applies to scalar conservation laws with convex fluxes, has been extended to systems of conservation laws where all the characteristic field are genuinely nonlinear, see for instance the works by Glimm and Lax [17], by Liu [22] and, more recently, by Bressan and Colombo [7], Bressan and Goatin [8] and Bressan and Yang [9], while for balance laws we refer to Christoforou and Trivisa [12].

2.1.1. *Finite time blow up of admissible solutions with large total variation.* Consider the general system of conservation laws

$$\partial_t U + \partial_x \big[F(U) \big] = 0, \tag{8}$$

where the unknown $U(t, x)$ attains values in $\mathbb{R}^N$, the variables $(t, x) \in [0, +\infty[\times\mathbb{R}$ and the flux function $F : \mathbb{R}^N \to \mathbb{R}^N$ is smooth and strictly hyperbolic (5). Consider furthermore the Cauchy problem obtained by coupling (8) with the initial condition

$$U(0, \cdot) = U_0. \tag{9}$$

Under some further technical assumption on the structure of the flux, Glimm [16] established existence of a global in time solution of the Cauchy problem provided that $\operatorname{TotVar} U_0$, the total variation of the initial datum, is sufficiently small. Under the same assumptions, Bressan and several collaborators established uniqueness results, see [5] for a detailed exposition.

The requirement that the total variation $\operatorname{TotVar} U_0$ is small is highly restrictive, but necessary to obtain well-posedness results unless additional assumptions are imposed on the flux function F. Indeed, explicit examples have been constructed of systems and data U_0 where $\operatorname{TotVar} U_0$ is finite, but large, and the admissible solution blows up in finite time. In particular, in [3] Baiti and Jenssen constructed an initial datum for system (1) such that the L^∞-norm of the admissible solution blows up in finite time. The solution is admissible in the sense that it is piecewise constant and every shock is Lax admissible. For further examples of finite time blow up, see the references in [3] and [14].

2.1.2. *Schaeffer's Regularity Theorem does not extend to systems.* In [24] Schaeffer established a regularity result which can be loosely speaking formulated as follows. Consider a scalar conservation law with strictly convex flux, namely equation (8) in the case when $U(t, x)$ attains real values and $F : \mathbb{R} \to \mathbb{R}$ is uniformly convex, i.e. $F'' \geq c > 0$ for some constant $c > 0$. The work by Kružkov [20] establishes existence and uniqueness of the so-called *entropy admissible solution* of the Cauchy problem posed by coupling (8) and (9). It is known that, even if U_0 is smooth, the entropy admissible solution can develop shocks, namely discontinuities that propagate in the (t, x)-plane. Schaeffer's Theorem states that, for a generic smooth initial datum, the number of shocks of the entropy admissible solution is locally finite. The word "generic" is here to be interpreted in a suitable technical sense, which is related to the Baire Category Theorem, see [24] for the precise statement.

In [11] we discuss whether or not Schaeffer's Theorem extends to systems of conservation laws where every characteristic field is genuinely nonlinear, namely (6) holds. Note that the assumption that every characteristic field is genuinely nonlinear can be loosely speaking regarded as the analogous for systems of the condition (which applies to scalar equations) that the flux is strictly convex. Indeed, regularity results for scalar equations with strictly convex fluxes have been extended to systems where every characteristic field is genuinely nonlinear: as we mentioned before, this is the case of Oleĭnik's [23] decay estimate, see for instance [7, 8, 9, 12, 17, 22] for possible extensions to systems. Also, the SBV regularity result by Ambrosio and De Lellis [1], which applies to scalar conservation laws with strictly convex fluxes, has been extended to systems where every characteristic field is genuinely nonlinear, see [2, 4, 13].

Despite the above considerations, in [11] we exhibit an explicit example which rules out the possibility of extending Schaeffer's Theorem to systems of conservation laws where every characteristic field is genuinely nonlinear. More precisely, we construct a "big" set of initial data such that the corresponding solutions of the Cauchy problems for the Baiti-Jenssen system (1) develop infinitely many shocks on a given compact set of the (t, x)-plane. The term "big" is to be again interpreted in a suitable technical sense, which is related to the Baire Category Theorem, see [11] for the technical details.

2.2. **The Lax solution of the Riemann problem.** We consider a system of conservation laws (8) and we assume that $F : \mathbb{R}^3 \to \mathbb{R}^3$ is strictly hyperbolic (5) and that every characteristic field is genuinely nonlinear, namely (6) holds for $i = 1, 2, 3$. Lemma 2.1 below states that the Baiti-Jenssen system satisfies these conditions. The Riemann problem is posed by coupling (8) with an initial datum in the form

$$U(0, x) := \begin{cases} U^- & x < 0 \\ U^+ & x > 0, \end{cases} \tag{10}$$

where U^+ and U^- are given states in $\mathbb{R}^3$. In [21], Lax constructed a solution of the Riemann problem (8)-(10) under the assumptions that the states U^+ and U^- are sufficiently close: we now briefly recall the key steps of the analysis in [21].

We fix $i = 1, 2, 3$ and $\bar{U} \in \mathbb{R}^3$ and we define the *i-wave fan curve* through $\bar{U}$ by setting

$$D_i[s, \bar{U}] := \begin{cases} R_i[s, \bar{U}] & s \geq 0 \\ S_i[s, \bar{U}] & s < 0 \end{cases} \tag{11}$$

In the previous expression, R_i is the *i-rarefaction curve* through $\bar{U}$ and S_i is the *i-Hugoniot locus* through $\bar{U}$. The i- rarefaction curve R_i is the integral curve of the vector field $\vec{r}_i$, namely the solution of the Cauchy problem

$$\begin{cases} \dfrac{dR_i}{ds} = \vec{r}_i(R_i) \\ \\ R_i[0, \bar{U}] = \bar{U}. \end{cases} \tag{12}$$

The i-th Hugoniot locus S_i is the set of states that can be joined to $\bar{U}$ by a shock with speed close to $\lambda_i(\bar{U})$. The i-Hugoniot locus S_i is determined by imposing the Rankine-Hugoniot conditions. We term the value $|s_i|$ *strength* of the i-wave connecting the states $\bar{U}$ (on the left) and $D_i[s, \bar{U}]$ (on the right). Note that, owing to (11), when $s_i > 0$ the i-wave is a i-th rarefaction wave, when $s_i < 0$ the i-wave is an i-shock satisfying the so-called *Lax admissibility criterion.* The solution of the Riemann problem (8)-(10) is computed by imposing

$$U^+ = D_3\Big[s_3, D_2\big[s_2, D_1[s_1, U^-]\big]\Big]$$

and by using the Local Invertibility Theorem to solve for (s_1, s_2, s_3). From the value of (s_1, s_2, s_3) one can reconstruct a solution of the Riemann problem (8)-(10), see [21] for the precise construction. This solution is obtained by patching together rarefaction waves and shocks that satisfy the Lax admissibility criterion. In the following, we refer to this solution as the *Lax solution* of the Riemann problem (8)-(10).

2.3. **The wave fan curves of the Baiti-Jenssen system.**

We collect in this paragraph some features of the Baiti-Jenssen system. For the proof, we refer to [3, 11].

The first result states that in the unit ball the Baiti-Jenssen system is strictly hyperbolic whenever $0 \leq \eta < 1/4$. Also, when $\eta > 0$ all the characteristic fields are genuinely nonlinear. Note that when $\eta = 0$ this last condition is lost because two characteristic fields became linearly degenerate. See [3] or [11] for the explicit computations.

Lemma 2.1. *Assume that $0 \leq \eta < 1/4$ and that U varies in the unit ball, $|U| < 1$. Then the Baiti-Jenssen system with flux (2) is strictly hyperbolic, namely (5) holds true. If we also have $\eta > 0$ then every characteristic field is genuinely nonlinear, namely (6) is satisfied for $i = 1, 2, 3$.*

We now discuss the structure of the wave fan curves. We start by giving the explicit expression of the 1- and the 3-wave fan curve. In the statement of the following result, we denote by $(\bar{u}, \bar{v}, \bar{w})$ the components of the state $\bar{U} \in \mathbb{R}^3$.

Lemma 2.2. *Consider the flux function (2), assume that $0 < \eta < 1/4$ and fix $\bar{U} \in \mathbb{R}^3$ such that $|\bar{U}| < 1$. Then the following properties hold true.*

i) The 1-wave fan curve $D_1[\sigma, \bar{U}]$ is a straight line in the plane $v = \bar{v}$, more precisely

$$D_1[\sigma, \bar{U}] = \bar{U} + \sigma \vec{r}_1(\bar{U}), \tag{13}$$

$$\textit{where} \quad \vec{r}_1(\bar{U}) = \begin{pmatrix} 1 \\ 0 \\ \bar{v} \end{pmatrix}.$$

Note that $\vec{r}_1(\bar U)$ is the first eigenvector of the Jacobian matrix $DF(\bar U)$. Also, the states $\bar U$ (on the left) and $D_1[\sigma, \bar U]$ (on the right) are connected by a wave which is

- *a 1-rarefaction wave when $\sigma > 0$,*
- *a Lax admissible 1-shock when $\sigma < 0$.*

ii) The 3-wave fan curve $D_3[\tau, \bar U]$ is a straight line in the plane $v = \bar v$, more precisely

$$D_3[\tau, \bar U] = \bar U + \tau \vec{r}_3(\bar U), \tag{14}$$

$$\text{where} \quad \vec{r}_3(\bar U) = \begin{pmatrix} 1 \\ 0 \\ \bar v - 2 \end{pmatrix}.$$

The vector $\vec{r}_3(\bar U)$ is the third eigenvector of the Jacobian matrix $DF(\bar U)$. Also, the states $\bar U$ (on the left) and $D_3[\tau, \bar U]$ (on the right) are connected by a wave which is

- *a 3-rarefaction wave when $\tau < 0$,*
- *a Lax admissible 3-shock when $\tau > 0$.*

Note that, for the 3-wave fan curve, the *positive* values of τ correspond to shocks, the *negative* values to rarefaction waves. This is the contrary with respect to (11) and it is a consequence of the fact that we use the same notation as in [3, 11] and we choose the orientation of $\vec{r}_3$ in such a way that when $\eta > 0$ condition (6) is replaced by the opposite inequality

$$\nabla \lambda_3 \cdot \vec{r}_3 < 0.$$

We now turn to the structure of the 2-wave fan curve. In the following statement, we use the notation

$$U^- = \begin{pmatrix} u^- \\ v^- \\ w^- \end{pmatrix}, \qquad U^+ = \begin{pmatrix} u^+ \\ v^+ \\ w^+ \end{pmatrix}.$$

Also, we consider *entropy admissible solutions* of scalar conservation laws, in the Kružkov [20] sense.

Lemma 2.3. *Assume that U is a Lax solution of the Riemann problem (8)-(10). Then the second component v is an entropy admissible solution of the Cauchy problem*

$$\begin{cases} \partial_t v + \partial_x [v^2] = 0 \\ v(0, x) = \begin{cases} v^- & x < 0 \\ v^+ & x > 0. \end{cases} \end{cases} \tag{15}$$

Also, we can choose the eigenvector $\vec{r}_2$ and the parametrization of the 2-wave fan curve $D_2[s, \bar U]$ in such a way that the second component of $D_2[s, \bar U]$ is exactly $\bar v + s$.

3. Interaction of two 2-shocks

We first rigorously state Lemma 1.1

Lemma 3.1. *There is a sufficiently small constant $\varepsilon > 0$ such that the following holds. Fix a constant a such that $0 < a < 1/2$ and set $U_\sharp := (a, 0, -a)$. Assume that*

$$|U_\ell - U_\sharp| \le \varepsilon a, \qquad 0 \le \eta \le \varepsilon a,$$

$$s_1, s_2 < 0, \qquad s_1, s_2 \in [-\varepsilon a, 0].$$

Assume furthermore that

$$U_r = D_2\Big[s_2, D_2[s_1, U_\ell]\Big]. \tag{16}$$

Then there are $\sigma < 0$ *and* $\tau > 0$ *such that*

$$U_r = D_3\Big[\tau, D_2\big[s_1 + s_2, D_1[\sigma, U_\ell]\big]\Big]. \tag{17}$$

Note that by combining (17) with the inequalities $\sigma < 0$, $\tau > 0$ and $s_1 + s_2 < 0$ we get that the three outgoing waves are all shocks. The proof of Lemma 3.1 is organized as follows:

- § 3.1: by relying on a perturbation argument, we show that the proof of Lemma 3.1 boils down to the proof of the Taylor expansion (21).
- § 3.2: we complete the proof by establishing (21).

3.1. **Proof of Lemma 3.1: first step.**

We start with some preliminary considerations. Assume that the states U_ℓ and U_r satisfy (16). Next, solve the Riemann problem between U_ℓ (on the left) and U_r (on the right): owing to [21], this amounts to determine by relying on the Local Invertibility Theorem the real numbers σ, s and τ such that

$$U_r = D_3\Big[\tau, D_2\big[s, D_1[\sigma, U_\ell]\big]\Big]. \tag{18}$$

Establishing the proof of Lemma 3.1 amounts to prove that $s = s_1 + s_2 < 0$ and that $\sigma < 0$, $\tau > 0$.

To prove that $s = s_1 + s_2$ we recall Lemma 2.3 and the fact that the v component is constant along the 1-st and the 3-rd wave fan curves D_1 and D_3. We conclude that $s = v_r - v_\ell = s_1 + s_2 < 0$. Note that v_r and v_ℓ are the second component of U_r and U_ℓ.

We are left to prove that $\sigma < 0$ and $\tau > 0$. We first introduce some notation: we regard σ and τ as functions of η, s_1 and s_2 and U_ℓ and we write $\sigma_\eta(s_1, s_2, U_\ell)$ and $\tau_\eta(s_1, s_2, U_\ell)$ to express this dependence. Note that σ and τ depend on η because the wave fan curve D_2 depends on η.

Owing to the Implicit Function Theorem, the regularity of $\sigma_\eta(s_1, s_2, U_\ell)$ and $\sigma_\eta(s_1, s_2, U_\ell)$ is at least the same as the regularity of the functions D_1, D_2 and D_3. Also, note that the Lax Theorem [21] (see also [5, p.101]) states that the wave fan curves D_1, D_2 and D_3 are C^2. The reason why we can achieve C^∞ regularity is because we are actually considering the wave fan curves in regions where they are C^∞. To see this, we first point out that, owing to (13) and (14), the wave fan curves D_1, D_3 are straight lines and hence they are C^∞. Next, we point out that we are only interested in *negative* values of $s_1 + s_2$. Hence, we can replace the 2-wave fan curve D_2 defined as in (11) with the 2-Hugoniot locus S_2. We recall that the 2-Hugoniot locus $S_2[s, \bar U]$ contains all the states that can be connected to $\bar U$ by a shock, namely all the states such that the couple $(\bar U, S_2[s, \bar U])$ satisfies the Rankine-Hugoniot conditions. The 2-Hugoniot locus $S_2[s, \bar U]$ is C^∞ and by combining all the previous observations we can conclude that $\sigma_\eta(s_1, s_2, U_\ell)$ and $\tau_\eta(s_1, s_2, U_\ell)$ are both C^∞ with respect to the variables (η, s_1, s_2, U_ℓ).

Next, we discuss the partial derivatives of $\sigma_\eta(s_1, s_2, U_\ell)$ and $\tau_\eta(s_1, s_2, U_\ell)$ with respect to (s_1, s_2) at the point $(\eta, 0, 0, U_\ell)$. By arguing as in the proof of estimate (7.32) in [5, p.133] we conclude that

- for every U_ℓ, for every $\eta > 0$ and every integer $k \geq 1$ we have the following equalities:

$$\left.\frac{\partial^k \sigma_\eta}{\partial s_1^k}\right|_{(0,0,U_\ell)} = \left.\frac{\partial^k \sigma_\eta}{\partial s_2^k}\right|_{(0,0,U_\ell)} = \left.\frac{\partial^k \tau_\eta}{\partial s_1^k}\right|_{(0,0,U_\ell)} = \left.\frac{\partial^k \tau_\eta}{\partial s_2^k}\right|_{(0,0,U_\ell)} = 0. \tag{19}$$

- For every U_ℓ and for every $\eta > 0$ we also have the following equality concerning the derivatives of second order:

$$\left.\frac{\partial^2 \sigma_\eta}{\partial s_1 \partial s_2}\right|_{(0,0,U_\ell)} = \left.\frac{\partial^2 \tau_\eta}{\partial s_1 \partial s_2}\right|_{(0,0,U_\ell)} = 0.$$

This implies that σ_η and τ_η admit the following Taylor expansions

$$\begin{aligned} \sigma_\eta(s_1, s_2, U_\ell) = & \frac{1}{2}\frac{\partial^3 \sigma_\eta(0,0,U_\ell)}{\partial s_1^2 \partial s_2} s_1^2 s_2 + \frac{1}{2}\frac{\partial^3 \sigma_\eta(0,0,U_\ell)}{\partial s_1 \partial s_2^2} s_1 s_2^2 \\ & + o(|(s_1,s_2)|)\, s_1 s_2 (s_1 + s_2) \\ \tau_\eta(s_1, s_2, U_\ell) = & \frac{1}{2}\frac{\partial^3 \tau_\eta(0,0,U_\ell)}{\partial s_1^2 \partial s_2} s_1^2 s_2 + \frac{1}{2}\frac{\partial^3 \tau_\eta(0,0,U_\ell)}{\partial s_1 \partial s_2^2} s_1 s_2^2 + \\ & + o(|(s_1,s_2)|)\, s_1 s_2 (s_1 + s_2) \end{aligned} \tag{20}$$

In § 3.2 we prove that when $\eta = 0$ and $U_\ell = U_\sharp$ the functions σ and τ admit the Taylor expansions

$$\begin{pmatrix} \sigma_0(s_1, s_2, U_\sharp) \\ \tau_0(s_1, s_2, U_\sharp) \end{pmatrix} = \frac{a}{32} \begin{pmatrix} 1 \\ -1 \end{pmatrix} s_1 s_2 (s_1 + s_2) + o(|(s_1, s_2)|^3). \tag{21a}$$

Next, we use the Lipschitz continuous dependence of the derivatives of third order with respect to η and U_ℓ and we conclude that

$$\begin{aligned} \left| \frac{\partial^3 \sigma_\eta(0,0,U_\ell)}{\partial s_1^2 \partial s_2} - \frac{a}{16} \right| + \left| \frac{\partial^3 \sigma_\eta(0,0,U_\ell)}{\partial s_1 \partial s_2^2} - \frac{a}{16} \right| < C\varepsilon a \\ \left| \frac{\partial^3 \tau_\eta(0,0,U_\ell)}{\partial s_1^2 \partial s_2} + \frac{a}{16} \right| + \left| \frac{\partial^3 \tau_\eta(0,0,U_\ell)}{\partial s_1 \partial s_2^2} + \frac{a}{16} \right| < C\varepsilon a \end{aligned}$$

provided that $0 \leq \eta \leq \varepsilon a$ and $|U_\ell - U_\sharp| \leq \varepsilon a$. In the above expression, C denotes a universal constant. By plugging the above expressions into (20) and recalling that $s_1, s_2 < 0$ we can eventually conclude that, if ε is sufficiently small, then

$$\begin{aligned} \sigma_\eta(s_1, s_2, U_\ell) < & \frac{a}{64} s_1 s_2 (s_1 + s_2) < 0, \\ \tau_\eta(s_1, s_2, U_\ell) > & -\frac{a}{64} s_1 s_2 (s_1 + s_2) > 0. \end{aligned}$$

The proof of the lemma is complete.

3.2. **Proof of formula (21).**

The proof of the Taylor expansion (21) is divided into two parts:

§ 3.2.1: as a preliminary result we determine the structure of the Hugoniot locus $S_2[s, \underline{U}]$

§ 3.2.2: we conclude the proof.

Note that in this paragraph we always assume $\eta = 0$ because formula (21) deals with this case.

3.2.1. *The 2-Hugoniot locus.* Before giving the technical results, we introduce some notation. First, we recall that we term F_0 the flux function F_η in (2) in the case when $\eta = 0$. In the following, we will mostly focus on the behavior of the first and the third component of U. Hence, it is convenient to term $\hat{U}$ and $\hat{F}_0$ the vectors obtained by erasing the second components of U and F_0, respectively. We have the relation

$$\widehat{F}_0(U) = 4\begin{pmatrix} v-1 & -1 \\ v(v-2) & 1-v \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = \widehat{J}(v)\cdot\widehat{U}, \tag{22}$$

where we have also introduced the 2×2 matrix $\widehat{J}(v)$.

Finally, we recall that we term $S_2[s,\bar U]$ the 2-Hugoniot locus passing through $\bar U$, namely the set of states that can be connected to $\bar U$ by a (possibly not admissible) shock of the second family. Also, as usual we denote by $\bar u$, $\bar v$ and $\bar w$ the first, second and third component of $\bar U$, respectively. We use the notation $\widehat{\bar U} = (\bar u, \bar w)$.

Lemma 3.2. *Fix $\eta = 0$ and assume that $|2\bar v + s| < 4$, then the 2-Hugoniot locus through $\bar U$ has the following expression: the second component of $S_2[s,\bar U]$ is $\bar v + s$ while the first and third components are*

$$\widehat{S_2}[s,\bar U] = \widehat{\bar U} + \boldsymbol{E}(\bar v, s)\widehat{\bar U} \tag{23}$$

where the 2×2 matrix $\boldsymbol{E}(\bar v, s)$ is

$$\boldsymbol{E}(\bar v, s) = \frac{4s}{(2\bar v + s)^2 - 16}\begin{pmatrix} s+4-2\bar v & 4 \\ (s+4)(s-2)+4\bar v & 3s-4+2\bar v \end{pmatrix}.$$

Proof. By Lemma 2.3 the second component of $S_2[s,\bar U]$ is $\bar v + s$. To construct $S_2[s,\bar U]$ we use the Rankine-Hugoniot conditions, which are a system of 3 equations. Owing to Lemma 2.3, the second equation reads

$$\gamma s = (\bar v + s)^2 - \bar v^2$$

and this implies that the speed γ of the 2-shock is

$$\gamma = 2\bar v + s. \tag{24}$$

We define the vector $\mathfrak{A}(s,\bar U)$ by setting

$$\mathfrak{A}(s,\bar U) := \widehat{S_2}[s,\bar U] - \widehat{\bar U}$$

and we point out that to establish Lemma 3.2 we are left to show that

$$\mathfrak{A}(s,\bar U) = \boldsymbol{E}(\bar v, s)\widehat{\bar U}. \tag{25}$$

The first and the third equations in the Rankine-Hugoniot conditions can be written as

$$\gamma\mathfrak{A}(s,\bar U) = \widehat{J}(\bar v + s)\left[\widehat{\bar U} + \mathfrak{A}(s,\bar U)\right] - \widehat{J}(\bar v)\widehat{\bar U}, \tag{26}$$

where $\widehat{J}$ is the same as in (22). Next, we introduce the 2×2 matrix

$$\begin{aligned} \boldsymbol{A}(v,\gamma) &= \gamma\boldsymbol{I} - \widehat{J}(v) \\ &= \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} - 4\begin{pmatrix} v-1 & -1 \\ v(v-2) & 1-v \end{pmatrix}, \end{aligned}$$

and we rewrite (26) as

$$\boldsymbol{A}\left(\bar v + s, \gamma\right)\mathfrak{A}(s,\bar U) = \left[\widehat{J}(\bar v + s) - \widehat{J}(\bar v)\right]\widehat{\bar U},$$

which implies (25) provided that

$$\boldsymbol{E}(\bar v, s) = \boldsymbol{A}^{-1}(\bar v + s, \gamma)\big[\widehat{J}(\bar v + s) - \widehat{J}(\bar v)\big]$$

By recalling that $\gamma = 2\bar v + s$ we can compute the explicit expression of the above matrices:

$$\begin{aligned}
\boldsymbol{A}(\bar v + s, 2\bar v + s) &= \begin{pmatrix} 4 - 3s - 2\bar v & 4 \\ 4(2 - \bar v - s)(\bar v + s) & 6\bar v + 5s - 4 \end{pmatrix}, \\
\widehat{J}(\bar v + s) - \widehat{J}(\bar v) &= 4s \begin{pmatrix} 1 & 0 \\ s + 2\bar v - 2 & -1 \end{pmatrix}.
\end{aligned}$$

The determinant of the matrix $\boldsymbol{A}(\bar v + s, 2\bar v + s)$ is

$$\underline{\mathrm{det}} := (2\bar v + s)^2 - 16$$

and hence the matrix is invertible when $|2\bar v + s| < 4$. We can now complete the lemma by computing the explicit expression of $\boldsymbol{E}$, namely

$$\begin{aligned}
\boldsymbol{E}(\bar v, s) &= \frac{1}{\underline{\mathrm{det}}} \begin{pmatrix} 6\bar v + 5s - 4 & -4 \\ 4(\bar v + s - 2)(\bar v + s) & 4 - 3s - 2\bar v \end{pmatrix} \cdot 4s \begin{pmatrix} 1 & 0 \\ s + 2\bar v - 2 & -1 \end{pmatrix} \\
&= \frac{4s}{\underline{\mathrm{det}}} \begin{pmatrix} s + 4 - 2\bar v & 4 \\ (s + 4)(s - 2) + 4\bar v & 3s - 4 + 2\bar v \end{pmatrix}.
\end{aligned}$$

□

3.2.2. *Conclusion of the proof of formula* (21). We are now ready to establish (21). We first recall some notation: we consider the system of conservation laws with flux F_0, see (2). We consider the collision between two 2-shocks and we assume that $U_\sharp = (a, 0, -a)$, U_m and U_r are the left, middle and right states before the interaction. This means that for some $s_1 < 0$, $s_2 < 0$ we have

$$\begin{aligned}
U_r &= D_2[s_2, U_m] = D_2\big[s_2, D_2[s_1, U_\sharp]\big] \\
&= S_2\big[s_2, S_2[s_1, U_\sharp]\big].
\end{aligned} \tag{27}$$

In the above expression, S_2 represents the 2-Hugoniot locus. To establish the last equality we used the fact that s_1 and s_2 are both negative. We plug (23) into (27) and we use the equality $v_\sharp = 0$: we arrive at

$$\begin{aligned}
\widehat{U_r} &= \Big[\widehat{U_\sharp} + \boldsymbol{E}(0, s_1)\widehat{U_\sharp}\Big] + \boldsymbol{E}(s_1, s_2)\Big[\widehat{U_\sharp} + \boldsymbol{E}(0, s_1)\widehat{U_\sharp}\Big] \\
&= \widehat{U_\sharp} + \Big[\boldsymbol{E}(0, s_1) + \boldsymbol{E}(s_1, s_2) + \boldsymbol{E}(s_1, s_2)\boldsymbol{E}(0, s_1)\Big]\widehat{U_\sharp}.
\end{aligned} \tag{28}$$

Next, we focus on the states after the interaction. By arguing as at the beginning of § 3.1, we conclude that it suffices to determine $\sigma = \sigma_0(s_1, s_2, U_\sharp)$ and $\tau = \tau_0(s_1, s_2, U_\sharp)$ such that

$$U_r = D_3\Big[\tau, D_2\big[s_1 + s_2, D_1[\sigma, U_\sharp]\big]\Big].$$

By the explicit expression of D_1 and D_3 and by applying Lemma 3.2 we infer that the above equality implies

$$\begin{aligned}
\widehat{U}_r &= \Big[\widehat{U_\sharp} + \sigma\widehat{\vec r_1}(0)\Big] + \boldsymbol{E}(0, s_1 + s_2)\Big[\widehat{U_\sharp} + \sigma\widehat{\vec r_1}(0)\Big] + \tau\widehat{\vec r_3}(s_1 + s_2) \\
&= \widehat{U_\sharp} + \boldsymbol{E}(0, s_1 + s_2)\widehat{U_\sharp} + \Big[\boldsymbol{I} + \boldsymbol{E}(0, s_1 + s_2)\Big]\sigma\widehat{\vec r_1}(0) + \tau\widehat{\vec r_3}(s_1 + s_2) \\
&= \widehat{U_\sharp} + \boldsymbol{E}(0, s_1 + s_2)\widehat{U_\sharp} + \mathbf{H}(s_1 + s_2)\begin{pmatrix} \sigma \\ \tau \end{pmatrix}.
\end{aligned} \tag{29}$$

In the previous expression we denote by $\widehat{\vec{r}_1}$ and $\widehat{\vec{r}_3}$ the vectors obtained from $\vec{r}_1$ and $\vec{r}_2$ by erasing the second component. Also, we introduced the matrix $\mathbf{H}$: its first column is $\big[\boldsymbol{I} + \boldsymbol{E}(0, s_1 + s_2)\big]\widehat{\vec{r}_1}(0)$, the second column is $\widehat{\vec{r}_3}(s_1 + s_2)$. In the following, we will prove that $\mathbf{H}(s_1 + s_2)$ is invertible provided that s_1 and s_2 are both sufficiently close to 0. By comparing (28) and (29) we then obtain

(30)

$$\begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \underbrace{\mathbf{H}^{-1}(s_1 + s_2)\Big[\boldsymbol{E}(0, s_1) + \boldsymbol{E}(s_1, s_2) + \boldsymbol{E}(s_1, s_2)\boldsymbol{E}(0, s_1) - \boldsymbol{E}(0, s_1 + s_2)\Big]}_{\boldsymbol{G}(s_1, s_2)}\widehat{U_\sharp}.$$

Assume that we have established the following asymptotic expansion for $\boldsymbol{G}$:

$$\boldsymbol{G}(s_1, s_2) = \frac{1}{32}\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix} s_1 s_2 (s_1 + s_2) + o(|(s_1, s_2)|^3). \tag{31}$$

Then by plugging both (31) and $\widehat{U_\sharp} = (a, -a)$ into (30) we obtain the asymptotic expansion (21). Hence, to conclude the proof of (21) we are left to establish (31).

First, we point out that, owing to the expression of $\boldsymbol{E}$ in the statement of Lemma 3.2,

$$\boldsymbol{E}(0, s) = \frac{4s}{s^2 - 16}\begin{pmatrix} s + 4 & 4 \\ (s + 4)(s - 2) & 3s - 4 \end{pmatrix}.$$

This implies that when $s_1 = s_2 = 0$, the matrix $\boldsymbol{E}(0, s_1 + s_2)$ vanishes and hence

$$\mathbf{H}^{-1}(0) = \Big(\hat{\vec{r}}_1(0)|\hat{\vec{r}}_3(0)\Big)^{-1} \overset{(13),(14)}{=} \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1/2 \\ 0 & -1/2 \end{pmatrix}.$$

We compute now the asymptotic expansion of

$$\boldsymbol{E}(0, s_1) + \boldsymbol{E}(0 + s_1, s_2) - \boldsymbol{E}(0, s_1 + s_2) + \boldsymbol{E}(0 + s_1, s_2)\boldsymbol{E}(0, s_1).$$

By directly computing the sum of the above matrices, we obtain that we can factor the term

$$\frac{4 s_1 s_2 (s_1 + s_2)}{(s_1^2 - 16)((s_1 + s_2)^2 - 16)((2s_1 + s_2)^2 - 16)},$$

which multiplies the matrix with coefficients

$$\begin{aligned}
&\text{Coeff}_{1,1}: && (s_1 + 4)(s_1 + s_2 + 4)(6s_1 + 5s_2 - 12) \\
&\text{Coeff}_{1,2}: && 4(5s_2^2 + 13s_1 s_2 + 9s_1^2 - 48) \\
&\text{Coeff}_{2,1}: && 2(s_1 + 4)(s_1 + s_2 + 4)(4 - 6s_1 \\
& && \quad + 2s_1^2 - 7s_2 + 4s_1 s_2 + 2s_2^2) \\
&\text{Coeff}_{2,2}: && 192 - 128s_1 - 36s_1^2 + 26s_1^3 \\
& && \quad - 160s_2 - 52s_1 s_2 + 65s_1^2 s_2 - 20s_2^2 \\
& && \quad + 55s_1 s_2^2 + 16s_2^3.
\end{aligned}$$

By combining the above computations we obtain the following asymptotic expansion:

$$\begin{aligned}
\boldsymbol{G}(s_1, s_2) &= -\frac{1}{4^5}\begin{pmatrix} 1 & 1/2 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} -3 \cdot 4^3 & -3 \cdot 4^3 \\ 2 \cdot 4^3 & 3 \cdot 4^3 \end{pmatrix} \cdot s_1 s_2 (s_1 + s_2) + o(\|(s_1, s_2)\|^3) \\
&= \frac{1}{32}\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix} s_1 s_2 (s_1 + s_2) + o(\|(s_1, s_2)\|^3). \qquad (32)
\end{aligned}$$

This establishes (31) and hence concludes the proof of (21). □

4. INTERACTION OF A 1-SHOCK AND A 2-SHOCK

We first rigorously state Lemma 1.2.

Lemma 4.1. *There is a sufficiently small constant* $\varepsilon > 0$ *such that if* $0 \leq \eta \leq \varepsilon$*, then the following holds. Assume that the states* U_ℓ*,* $U_r \in \mathbb{R}^3$ *satisfy*

$$U_r = D_1\Big[\sigma, D_2[s, U_\ell]\Big] \tag{33}$$

for real numbers s*,* σ *such that*

$$\sigma, s < 0, \quad |s|, \ |\sigma| < \frac{1}{4}.$$

Furthermore, assume that $|U_\ell| < 1/2$*. Then there are real numbers* σ' *and* τ' *such that*

$$U_r = D_3\Big[\tau', D_2\big[s, D_1[\sigma', U_\ell]\big]\Big] \tag{34}$$

and

$$2\sigma \leq \sigma' \leq \frac{1}{2}\sigma, \qquad \frac{1}{100}\sigma s \leq \tau' \leq 10\sigma s. \tag{35}$$

Note that (35) implies $\sigma' < 0$ and $\tau' > 0$. If we combine these inequalities with (34) and $s < 0$ we see that the three outgoing waves are all shocks.

Establishing the proof of Lemma 4.1 amounts to establish (35). Indeed,

1. by using Lax's construction (see § 2.2) we determine σ', s', τ' such that
$$U_r = D_3\Big[\tau', D_2\big[s', D_1[\sigma', U_\ell]\big]\Big].$$
2. By combining (13), (14) and Lemma 2.3 we obtain that $s' = s$.

To establish (35) we proceed as follows:

- § 4.1: we establish (35) in the case when $\eta = 0$.
- § 4.2: we conclude the proof by relying on a perturbation argument. More precisely, by using the fact that the flux F_η in (2) smoothly depends on η we show that (35) holds provided η is sufficiently small.

Note that, as we have mentioned in the introduction, the precise expression of the function p_1 and p_3 plays no role in the proof of Lemma 4.1, what is actually relevant is that estimates (35) hold at $\eta = 0$ with strict inequalities and that η is sufficiently small.

4.1. **Proof of Lemma 4.1: the case** $\eta = 0$**.**

We establish (35) in the case $\eta = 0$. This part of the proof is actually the same as in [3, p. 844-845], but for completeness we go over the main steps.

We term σ'_0, τ'_0 the real numbers satisfying (34) when $\eta = 0$. Let v_r and v_ℓ denote the second components of U_r and U_ℓ, respectively. We term γ the speed of the incoming 2-shock (which is the same as the speed of the outgoing 2-shock), we recall Lemma 2.3 and the fact that the second component varies only across 2-shocks. We conclude that

$$\gamma = \frac{v_r^2 - v_\ell^2}{v_r - v_\ell} = v_r + v_\ell = 2v_\ell + s.$$

Since by assumption $|U_\ell| < 1/2$ and $|s| < 1/4$, then

$$|\gamma| < 3. \tag{36}$$

By imposing the Rankine-Hugoniot conditions on the incoming and outgoing 2-shocks and by arguing as in [3, pp. 844-845], with the choice $c = 4$, we arrive at the following system:

$$\begin{cases} (\gamma+4)\sigma_0' + (\gamma-4)\tau_0' = (\gamma+4)\sigma \\ v_\ell(\gamma+4)\sigma_0' + (v_\ell+s-2)(\gamma-4)\tau_0' = (v_\ell+s)(\gamma+4)\sigma \end{cases}$$

If we set

$$A := \begin{pmatrix} \gamma+4 & \gamma-4 \\ v_\ell(\gamma+4) & (v_\ell+s-2)(\gamma-4) \end{pmatrix} \tag{37}$$

and

$$X_0 = \begin{pmatrix} \sigma_0' \\ \tau_0' \end{pmatrix}, \quad Y = \begin{pmatrix} \gamma+4 \\ (v_\ell+s)(\gamma+4) \end{pmatrix}\sigma, \tag{38}$$

then the above linear system can be recast as $AX_0 = Y$. The explicit expression of the matrix A^{-1} is

$$\frac{1}{(4^2-\gamma^2)(-s+2)} \begin{pmatrix} (v_\ell+s-2)(\gamma-4) & -(\gamma-4) \\ -v_\ell(\gamma+4) & (\gamma+4) \end{pmatrix} \tag{39}$$

We solve for σ_0' and τ_0' and we obtain

$$\sigma_0' = \frac{2}{-s+2}\sigma, \qquad \tau_0' = \frac{\gamma+4}{(4-\gamma)(-s+2)}\, s\sigma \tag{40}$$

By using (36) and the inequality $|s| < 1/4$, we obtain

$$\frac{2}{3} < \frac{2}{-s+2} < 1, \quad \frac{1}{21} < \frac{\gamma+4}{(4-\gamma)(-s+2)} < 4 \tag{41}$$

and this implies that the estimate (35) holds true in the case when $\eta = 0$.

4.2. **Proof of Lemma 4.1: the case $\eta > 0$.** We are now ready to complete the proof of Lemma 4.1. We proceed as follows:

§ 4.2.1: we make some preliminary considerations which reduce the proof of Lemma 4.1 to the proof of the fact that a certain map is a strict contraction.

§ 4.2.2: we conclude the proof by showing that the map is indeed a strict contraction.

4.2.1. *Preliminary considerations.* We first introduce some notation. We term U_m the intermediate state *before* the interaction, namely

$$U_m := D_2[s, U_\ell]. \tag{42}$$

Also, we term U_m' and U_m'' the intermediate states *after* the interaction, namely

$$\begin{aligned} U_m' &:= D_1[\sigma', U_\ell], \\ U_m'' &:= D_2[s, U_m'] = D_3\big[-\tau', U_3\big] = D_3\big[-\tau', D_1[\sigma, U_m]\big] \\ &= D_3\big[-\tau', D_1[\sigma, D_2[s, U_\ell]]\big] \end{aligned} \tag{43}$$

Next, we use [3, eq. (5.3)-(5.4)] and we recast the Rankine-Hugoniot conditions for 2-shocks as a nonlinear system in the form

$$AX + \eta\mathcal{F}(X, U_\ell, s, \sigma) = Y, \tag{44}$$

where A and Y are as in (37) and (38), respectively. Also, the vector X is defined by setting

$$X := \begin{pmatrix} \sigma' \\ \tau' \end{pmatrix}$$

and the nonlinear term $\mathcal{F}(X, U_\ell, s, \sigma)$ is equal to

$$\begin{pmatrix} p_1(U''_m) - p_1(U_m) - p_1(U'_m) + p_1(U_\ell) \\ p_3(U''_m) - p_3(U_m) - p_3(U'_m) + p_3(U_\ell), \end{pmatrix}. \tag{45}$$

In the above expression, the functions p_1 and p_3 are the same as in (3). Note, however, that the precise expression of p_1 and p_3 plays no role in the proof, the only relevant point is that p_1 and p_3 are both regular (say twice differentiable with Lipschitz continuous second derivatives). Note furthermore that we can regard $\mathcal{F}$ as a function of X, U_ℓ, s and σ because, owing to (42) and (43), U_m, U'_m and U''_m are functions of X, U_ℓ, s and σ. Next, we rewrite equation (44) as

$$X = X_0 - \eta A^{-1} \mathcal{F}(X, U_\ell, s, \sigma), \tag{46}$$

where the vector $X_0 = A^{-1} Y$ is given by (38) and (40).

We now fix s, σ, η and $|U_\ell|$ satisfying the assumptions of Lemma 4.1 and we define the closed ball

$$\mathfrak{K} := \Big\{ X = (\sigma', \tau') \in \mathbb{R}^2 : \, |X - X_0| \leq k \eta \sigma s \Big\}. \tag{47}$$

In the above expression, $k > 0$ is a universal constant that will be determined in the following and σ'_0 and τ'_0 are defined by (40). We also define the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$T(X) := X_0 - \eta A^{-1} \mathcal{F}(X, U_\ell, s, \sigma). \tag{48}$$

Assume that T is a strict contraction from $\mathfrak{K}$ to $\mathfrak{K}$. Then the proof of Lemma 4.1 is complete: indeed, owing to (46) the fixed point X satisfies the Rankine-Hugoniot conditions (44). Also, owing to (41) and to (47) we infer that the inequalities (35) are satisfied provided that the parameter η is sufficiently small.

4.2.2. *Conclusion of the proof of Lemma 4.1.* In this paragraph we prove that the map T defined by (48) is a strict contraction on the closed set $\mathfrak{K}$ defined by (47).

First, we make some remarks about notation. To simplify the exposition, in the following we denote by C a universal constant: its precise value can vary from occurrence to occurrence. Also, in the following we will determine the constant k in (47) and then choose the constant η in such a way that $k\eta \leq 1$. This choice implies in particular that, when X belongs to the set $\mathfrak{K}$ defined as in (47) and the hypotheses of Lemma 4.1 are satisfied, then the map $\mathcal{F}$ attains values on a bounded set and so $\mathcal{F}$ and all its derivatives are bounded by some constant C. Finally, note that, if $X \in \mathfrak{K}$ and the hypotheses of Lemma 4.1 are satisfied, then $|A^{-1}| \leq C$.

We now proceed according to the following steps.

STEP 1: we point out that to show that the map T is a contraction it suffices to show that

$$|\mathcal{F}(X, U_\ell, s, \sigma)| \leq C \sigma s \tag{49}$$

provided that $X \in \mathfrak{K}$ and the hypotheses of Lemma 4.1 hold. Indeed, assume that (49) holds, then

$$|T(X) - X_0| \overset{(48)}{\leq} \eta |A^{-1} \mathcal{F}(X, U_\ell, s, \sigma)| \overset{(49)}{\leq} C \eta \sigma s$$

and hence T attains values in the set $\mathfrak{K}$ defined as in (47) provided that k is large enough. Also,

$$\begin{aligned} |T(X_1) - T(X_2)| &\overset{(48)}{\leq} \eta |A^{-1}| |\mathcal{F}(X_1, U_\ell, s, \sigma) - \mathcal{F}(X_2, U_\ell, s, \sigma)| \\ &\leq \eta C |X_1 - X_2| \leq \frac{1}{2} |X_1 - X_2| \end{aligned}$$

provided that the constant η is sufficiently small. This implies that T is a contraction and concludes the proof of Lemma 4.1.

STEP 2: we establish (49). First, we point out that, if $X \in \mathfrak{K}$ and the hypotheses of Lemma 4.1 are satisfied, then

$$\begin{aligned} |p_1(U''_m) - p_1(U_m) - p_1(U'_m) + p_1(U_\ell)| &\leq C\Big(|U''_m - U_m| + |U'_m - U_\ell| \Big) \\ &\leq \quad C\Big(|U''_m - U_r| + |U_r - U_m| + |U'_m - U_\ell| \Big) \\ &\leq \quad C\Big(|\tau'| + |\sigma| + |\sigma'| \Big) \leq C\Big(|\sigma| + |X'_0| + |X - X'_0| \Big) \\ &\overset{(40),(47)}{\leq} C\Big(|\sigma| + \eta k \sigma s \Big) \leq C|\sigma|. \end{aligned}$$

By using an analogous argument, we control the second component of $\mathcal{F}$ and we arrive at

$$|\mathcal{F}(X, U_\ell, s, \sigma)| \leq C|\sigma|. \tag{50}$$

Next, we point out that when $s = 0$ we have $U'_m = U''_m$ and $U_\ell = U_m$ and by using again the Lipschitz continuity of the functions p_1 and p_3 we conclude that

$$|\mathcal{F}(X, U_\ell, s, \sigma)| \leq C|s|. \tag{51}$$

Finally, we use the regularity of the function $\mathcal{F}$ and, by arguing as in the proof of [5, Lemma 2.5, p. 28], we combine (50) and (51) to obtain (49). This concludes the proof of Lemma 4.1. □

ACKNOWLEDGMENTS

The authors wish to thank the anonymous referees for their useful remarks that helped improve the exposition and allowed to shorten and improve the proof of Lemma 4.1. The second author wish to thank the organizers of the conference "Contemporary topics in Conservation Laws" for the invitation to give a talk and present results related to the topic of the present paper. Both authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and are supported by the PRIN national project "Nonlinear Hyperbolic Partial Differential Equations, Dispersive and Transport Equations: theoretical and applicative aspects". This work was also supported by the EPSRC Science and Innovation award to the OxPDE (EP/E035027/1).

REFERENCES

[1] L. Ambrosio and C. De Lellis, A note on admissible solutions of 1D scalar conservation laws and 2D Hamilton-Jacobi equations, *J. Hyperbolic Differ. Equ.*, **1** (2004), 813–826, URL `http://dx.doi.org/10.1142/S0219891604000263`.

[2] F. Ancona and K. T. Nguyen, *in preparation*.

[3] P. Baiti and H. K. Jenssen, Blowup in L^∞ for a class of genuinely nonlinear hyperbolic systems of conservation laws, *Discrete Contin. Dynam. Systems*, **7** (2001), 837–853, URL `http://dx.doi.org/10.3934/dcds.2001.7.837`.

[4] S. Bianchini and L. Caravenna, SBV regularity for genuinely nonlinear, strictly hyperbolic systems of conservation laws in one space dimension, *Comm. Math. Phys.*, **313** (2012), 1–33, URL `http://dx.doi.org/10.1007/s00220-012-1480-5`.

[5] A. Bressan, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, vol. 20 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2000.

[6] A. Bressan and G. M. Coclite, On the boundary control of systems of conservation laws, *SIAM J. Control Optim.*, **41** (2002), 607–622 (electronic), URL `http://dx.doi.org/10.1137/S0363012901392529`.

[7] A. Bressan and R. M. Colombo, Decay of positive waves in nonlinear systems of conservation laws, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **26** (1998), 133–160, URL `http://www.numdam.org/item?id=ASNSP_1998_4_26_1_133_0`.

[8] A. Bressan and P. Goatin, Oleinik type estimates and uniqueness for $n \times n$ conservation laws, *J. Differential Equations*, **156** (1999), 26–49, URL `http://dx.doi.org/10.1006/jdeq.1998.3606`.

[9] A. Bressan and T. Yang, A sharp decay estimate for positive nonlinear waves, *SIAM J. Math. Anal.*, **36** (2004), 659–677 (electronic), URL `http://dx.doi.org/10.1137/S0036141003427774`.

[10] L. Caravenna, A note on regularity and failure of regularity for systems of conservation laws via Lagrangian formulation, *Bull. Braz. Math. Soc. (N.S.), to appear. Also arXiv:1505.00531.*

[11] L. Caravenna and L. V. Spinolo, Schaeffer's Regularity Theorem for scalar conservation laws does not extend to systems, *preprint. Also arXiv:1505.00609.*

[12] C. Christoforou and K. Trivisa, Sharp decay estimates for hyperbolic balance laws, *J. Differential Equations*, **247** (2009), 401–423, URL `http://dx.doi.org/10.1016/j.jde.2009.03.013`.

[13] C. M. Dafermos, Wave fans are special, *Acta Math. Appl. Sin. Engl. Ser.*, **24** (2008), 369–374, URL `http://dx.doi.org/10.1007/s10255-008-8010-4`.

[14] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, vol. 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 3rd edition, Springer-Verlag, Berlin, 2010, URL `http://dx.doi.org/10.1007/978-3-642-04048-1`.

[15] R. J. DiPerna, Global solutions to a class of nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.*, **26** (1973), 1–28.

[16] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.*, **18** (1965), 697–715.

[17] J. Glimm and P. D. Lax, *Decay of solutions of systems of nonlinear hyperbolic conservation laws*, Memoirs of the American Mathematical Society, No. 101, American Mathematical Society, Providence, R.I., 1970.

[18] H. Holden and N. H. Risebro, *Front tracking for hyperbolic conservation laws*, vol. 152 of Applied Mathematical Sciences, Springer-Verlag, New York, 2002, URL `http://dx.doi.org/10.1007/978-3-642-56139-9`.

[19] H. K. Jenssen, Blowup for systems of conservation laws, *SIAM J. Math. Anal.*, **31** (2000), 894–908.

[20] S. N. Kružkov, First order quasilinear equations with several independent variables., *Mat. Sb. (N.S.)*, **81 (123)** (1970), 228–255.

[21] P. D. Lax, Hyperbolic systems of conservation laws. II, *Comm. Pure Appl. Math.*, **10** (1957), 537–566.

[22] T. P. Liu, Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.*, **30** (1977), 586–611.

[23] O. A. Oleĭnik, Discontinuous solutions of non-linear differential equations, *Uspehi Mat. Nauk (N.S.)*, **12** (1957), 3–73.

[24] D. G. Schaeffer, A regularity theorem for conservation laws, *Advances in Math.*, **11** (1973), 368–386.

E-mail address: `laura.caravenna@math.unipd.it`
E-mail address: `spinolo@imati.cnr.it`